HOLDER CONTINUITY OF THE GRADIENT OF THE SOLUTIONS OF CERTAIN DEGENERATE PARABOLIC EQUATIONS

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Abstract

This paper is concerned with the parabolic equation
\[ \frac{\partial u}{\partial t} - \text{div}(|V\nabla u|^{p-2}V\nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^n, \ t > 0 \]
with \( p > \max \left\{ \frac{3}{2}, \frac{2N}{N+2} \right\} \) which is degenerate if \( p < 2 \) or singular if \( \frac{3}{2} < p < 2 \). Let \( u(x, t) \) be any weak solution of the equation in \( L^\infty[0, T; L^2(\Omega)] \cap L^p[0, T; W^{1,p}(\Omega)] \). The Hölder continuity of \( V\nabla u \) is established.

§1. Introduction

In this paper we consider the parabolic equation
\[ \frac{\partial u}{\partial t} - \text{div}(|V\nabla u|^{p-2}V\nabla u) = 0 \] (1.1)
with \( p > \max \left\{ \frac{3}{2}, \frac{2N}{N+2} \right\} \), where \( \nabla = \text{grad} \) and \( x \) varies in an open set \( \Omega \subset \mathbb{R}^n \). The equation is degenerate if \( p > 2 \) or singular if \( \frac{3}{2} < p < 2 \).

By a solution \( u \) of (1.1) in the cylindrical domain \( \Omega_T = \Omega \times (0, T] \) we mean a function, \( u(x, t) \), defined in \( \Omega_T \) and satisfying:
\[ u \in L^\infty[0, T; L^2(\Omega)] \cap L^p[0, T; W^{1,p}(\Omega)] \] (1.2)
and (1.1) holds in the weak sense
\[ \int_{\Omega_T} [u \varphi_t - |V\nabla u|^{p-2}V\nabla u \cdot \nabla \varphi] \, dx \, dt = 0, \quad \forall \varphi \in C_0^1(\Omega_T). \] (1.3)

The main result of this paper asserts that for any solution \( u \) of (1.1) with
\[ \max \left\{ \frac{3}{2}, \frac{2N}{N+2} \right\} < p < \infty, \]
\( V\nabla u \) is local Hölder continuity in \( \Omega_T \). For any \( \Omega'_T \subset \subset \Omega_T \), the Hölder coefficient and exponent of \( V\nabla u \) depend only on \( \Omega'_T \) and on a bound on the norm of \( u \) with respect...
to the spaces in (1.2).

For the equation (1.1), N. D. Alikakos and L. C. Evans obtained the continuity of $\nabla u$ with $p>2$. They applied Evan's method for elliptic equations in [2] to (1.1), but they could not manage non-uniformity between space and time parts very well and so they had not found Hölder continuity of $\nabla u$. E. Dibenedetto and A. Friedman studied the corresponding parabolic systems, which contain the equation (1.1) as a special case, but they established only the continuity of $\nabla u$ with modulus $\omega(r) = \left(\log \log \frac{A}{r}\right)^{-\sigma}$ for some constants $A$ and $\sigma$. They asserted that their results hold for $p>\max\left\{1, \frac{2N}{N+2}\right\}$, but they made some mistake in the calculation.

The coefficient of the first term on the right hand side of the inequality (iii) on p. 87 in [3] should be $(p-2)/4$ instead of $(p-2)/2$ and, in fact, their results hold only for $p>\max\left\{\frac{3}{2}, \frac{2N}{N+2}\right\}$.

The paper is organized as follows:

In § 2 we establish $L^p$ estimate on $\nabla u$. E. Dibenedetto and A. Friedman emphasized that both the Moser and De Giorgi iteration on procedures must be used for it. In this paper we use only Moser iteration and so the proof is much shorter.

In § 3 we give some fundamental lemmas and finally we come to our conclusion in § 4.

§ 2. Boundedness of $\nabla u$

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ and set $\Omega_T = \Omega \times (0, T]$, $T>0$. Let $u$ be a solution of

$$\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega_T,$$

where $p>\max\left\{\frac{3}{2}, \frac{2N}{N+2}\right\}$, and set

$$\|u\|_p = \text{ess sup} \|u(\cdot, t)\|_{L^p, \Omega_T}$$

(2.2)

where $\|\cdot\|_{L^p, \Omega_T}$ is the norm in $L^p(\Omega_T)$. By definition we have $\|u\|_p < \infty$. For any $s>0$, set

$$\Omega_{T, s} = \Omega_s \times (s, T],$$

where

$$\Omega_s = \{x \in \Omega : \text{dist}\{x, \partial \Omega\} > s\}.$$

**Theorem 2.1.** For any $s>0$ and $T>0$ there exists a constant $C^*$, depending only on $N$, $p$, $s$, $T$ and $\Gamma$, such that, for any solution $u$ of (2.1) with $\|u\|_p < \Gamma$,

$$\|\nabla u\|_{L^p, \Omega_{T, s}} < C^*.$$  

(2.3)

**Proof** Fix any point $P_0(x^0, t^0) \in \Omega_T$ and set
for $R$ such that $Q(p_0, R) \subseteq \Omega_R$. For clarity we shall first establish (2.8) under the
assumption that $u, u_t, \nabla u, D_\alpha u$ are in suitable $L^p$ spaces so that the calculations
below are justified.

Differentiating (2.1) with respect to $x_j$ we get

$$
\frac{\partial}{\partial t} u_{x_j} - \text{div} \left( \left| \nabla u \right|^{p-2} \nabla u_{x_j} + \frac{\partial}{\partial x_j} \left| \nabla u \right|^{p-2} \nabla u \right) = u. \quad (2.4)
$$

Set $\phi = u_{x_j} v^{\alpha^2}$ and apply to (2.4) the test function

$$
\varphi = u_{x_j} v^{\alpha^2} \left( \alpha \geq \min \left\{ 0, \frac{p-2}{2} \right\} \right),
$$

where $\zeta$ is the usual $C^1$ cut-off function with respect to $Q(p_0, R), Q(p_0, (1-\sigma)R)$
$(0 < \sigma < 1)$. After some computation (cf. p. 87 in [3]) we get

$$
\frac{1}{2(\alpha+1)} \text{ess sup} \int_{t - \epsilon < t < t + \epsilon} \xi_3^2 v^{\alpha + 1} \, dx \, dt + \frac{p+2\alpha-2}{4} \int \xi_3^2 v^{\frac{p+2\alpha-4}{2}} \left| \nabla \zeta \right|^2 \, dx \, dt
$$

$$
+ \int \xi_3^2 v^{\frac{p+2\alpha-2}{2}} \left| D^2 u \right|^2 \, dx \, dt + \frac{\alpha(p-2)}{2} \int \xi_3^2 v^{\frac{p+2\alpha-4}{2}} \left( \nabla u \cdot \nabla \zeta \right) \, dx \, dt
$$

$$
\leq C \left( \int \xi v^{\frac{p+2\alpha-2}{2}} \left| \nabla \zeta \right| \left| \nabla \zeta \right| \, dx \, dt + \frac{2}{2(\alpha+1)} \int \xi_3^2 v^{\alpha + 1} \, dx \, dt \right). \quad \text{(2.5)}
$$

where $B(R) = B(x_0, R), Q(R) = Q(p_0, R)$. Here and after we always express
constants depending on $N$ and $p$ by $C$.

Noting that

$$
\varphi = u_{x_j} v^{\alpha^2} \left( \alpha \geq \min \left\{ 0, \frac{p-2}{2} \right\} \right),
$$

we obtain from (2.5)

$$
\frac{1}{2(\alpha+1)} \text{ess sup} \int_{t - \epsilon < t < t + \epsilon} \xi_3^2 v^{\alpha + 1} \, dx \, dt + \frac{p+2\alpha-2}{4} \int \xi_3^2 v^{\frac{p+2\alpha-4}{2}} \left| \nabla \zeta \right|^2 \, dx \, dt
$$

$$
+ \frac{\alpha(p-2)}{2} \int \xi_3^2 v^{\frac{p+2\alpha-2}{2}} \left( \nabla u \cdot \nabla \zeta \right) \, dx \, dt
$$

$$
\leq C \left( \int \xi v^{\frac{p+2\alpha-2}{2}} \left| \nabla \zeta \right| \left| \nabla \zeta \right| \, dx \, dt + \frac{2}{2(\alpha+1)} \int \xi_3^2 v^{\alpha + 1} \, dx \, dt \right). \quad \text{(2.7)}
$$

If $p+2\alpha-1 > 0$ and $\alpha(p-2) > 0$, we have, applying Cauchy inequality,

$$
\frac{1}{2(\alpha+1)} \text{ess sup} \int_{t - \epsilon < t < t + \epsilon} \xi_3^2 v^{\alpha + 1} \, dx \, dt + \frac{p+2\alpha-1}{4} \int \xi_3^2 v^{\frac{p+2\alpha-4}{2}} \left| \nabla \zeta \right|^2 \, dx \, dt
$$

$$
\leq \frac{C}{p+2\alpha-1} \int \xi v^{\frac{p+2\alpha-2}{2}} \left| \nabla \zeta \right|^2 \, dx \, dt + \frac{1}{2(\alpha+1)} \int \xi_3^2 v^{\alpha + 1} \, dx \, dt. \quad \text{(2.8)}
$$

We consider two cases:
Let $a>0$. Then it follows from (2.8) that, for any $a>0$,
\[
\text{ess sup}_{t-a^*<t<e} \int_{Q(R)} v^{p+2a} |\nabla (u^{4a})|^2 dx dt \\
\leq C \left[ \int_{Q(R)} \int_{Q(R-a^*)} v^{p+2a} dx dt + \int_{Q(R)} |\zeta_t| v^{a+1} dx dt \right].
\]  
(2.9)

where $C$ is independent of $a$.

Using Hölder inequality and Sobolev embedding theorem, we have
\[
\int_{Q(R-a^*)} v^{p+2a} dx dt \leq \left( \int_{Q(R-a^*)} v^{\frac{p+2a}{2}} dx \right)^{\frac{2}{p+2a}} \left( \int_{Q(R-a^*)} v^{\frac{p+2a}{2}} dx \right)^{\frac{p}{p+2a}} \\
\leq \text{ess sup}_{t-a^*<t<e} \left( \int_{Q(R-a^*)} v^{a+1} dx \right)^{\frac{2}{p+2a}} \int_{Q(R-a^*)} \left( v^{p+2a} + |\nabla (u^{4a})|^2 \right) dx dt \\
\leq C \left[ \int_{Q(R)} \int_{Q(R-a^*)} v^{p+2a} dx dt + \int_{Q(R)} |\zeta_t| v^{a+1} dx dt \right].
\]  
(2.10)

Set $\alpha=1+\frac{2}{N}$ and, for $l=0, 1, 2, \ldots$,
\[
R_l=R \left( \frac{1}{2} + \frac{1}{2^{l+1}} \right),
\]
\[
\zeta_l(x) \in C^0_0(Q(R_l)), 0 \leq \zeta_l(x) \leq 1, \zeta_l(x) \equiv 1 \text{ in } Q(R_{l+1}), \\
\alpha+l+\alpha'.
\]  
(2.11)

Then it follows from (2.10) that
\[
\left( \int_{Q(R_{l+1})} v^{\frac{p-2+\alpha'}{2}} dx dt \right)^{\frac{1}{\alpha'}} \leq C \frac{4l}{R} \left[ \int_{Q(R_l)} \int_{Q(R_{l+1})} v^{\frac{p-2+\alpha'}{2}} dx dt + \int_{Q(R_l)} v^\alpha dx dt \right].
\]  
(2.12)

Without loss of generality, we can assume that, for any $l \geq 0$,
\[
\int_{Q(R_l)} v^{\frac{p-2+\alpha'}{2}} dx dt \geq 1,
\]
and so from (2.12) we have, noting $p>2$,
\[
\left( \int_{Q(R_{l+1})} v^{\frac{p-2+\alpha'}{2}} dx dt \right)^{\frac{1}{\alpha'}} \leq C \frac{4l}{R} \left( \int_{Q(R_l)} v^{\frac{p-2+\alpha'}{2}} dx dt \right).
\]

The standard Moser iteration procedure yields
\[
\left( \int_{Q(R_{l+1})} v^{\frac{p-2+\alpha'}{2}} dx dt \right)^{\frac{1}{\alpha'}} \leq C \frac{4l}{R^{N+2}} \int_{Q(R_l)} v^{\frac{p-2}{2}} dx dt, \quad l=0, 1, 2, \ldots
\]

Let $l \to \infty$, we have
\[
\text{ess sup}_{Q(R_0)} v \leq \frac{C}{R^{N+2}} \int_{Q(R_0)} |\nabla u|^2 dx dt.
\]  
(2.18)
(2) $2 > p > \max \left\{ \frac{3}{2}, \frac{2N}{N+2} \right\}$.

Let $\alpha = \frac{p-2}{2}$. Obviously we have $p+2\alpha-1 > 0$. When $\alpha < 0$, (2.8) holds since $\alpha(p-2) > 0$. When $\alpha > 0$, the third term of (2.7) can be bounded below as

$$\frac{\alpha(p-2)}{2} \int \int \xi^2 v^{p+2\alpha-4} \left( \nabla u \cdot \nabla v \right)^2 \, dx \, dt \geq - \frac{\alpha(2-p)}{2} \int \int \xi^2 v^{p+2\alpha-4} |\nabla v|^2 \, dx \, dt.$$

It follows from (2.7) that, for $\alpha > 0$,

$$\frac{1}{2(\alpha+1)} \text{ess sup} \int \int \xi v^{p+2\alpha-2} \, dx \, dt \geq \frac{(1+2\alpha)(p-1)}{4} \int \int \xi^2 v^{p+2\alpha-4} |\nabla v|^2 \, dx \, dt$$

$$\leq C \int \int \xi v^{p+2\alpha-2} |\nabla \xi| |\nabla v| \, dx \, dt + \frac{1}{2(\alpha+1)} \int \int |\xi|^{p+2\alpha} \, dx \, dt.$$

Similarly, (2.9) and then (2.10) hold for $\alpha > N(p-2)$ and $2 > p > \frac{3}{2}$. Now set

$$\alpha_0 = \frac{N(2-p)}{4}, \lambda = 1 - \alpha_0 + \frac{p-2}{2}, \alpha + 1 = \alpha_0 + \lambda \beta,$$

where $\beta = 1 + \frac{2}{N}$. Instead of (2.12) we have

$$\left( \int \int \xi v^{p+2\alpha+2\lambda} \, dx \, dt \right)^{1/2} \leq C 4^t \left[ \int \int \xi^{p+2\alpha+2\lambda} \, dx \, dt + \int \int \xi^{p+\lambda\beta} \, dx \, dt \right].$$

Clearly, $p > \frac{2N}{N+2}$ implies $\lambda > 0$. Without loss of generality, we can assume

$$\int \int \xi v^{p+\lambda\beta} \, dx \, dt > 1$$

for any $t > 0$.

and then we have, noting $p < 2$,

$$\left( \int \int \xi v^{p+2\alpha+2\lambda} \, dx \, dt \right)^{1/2} \leq C 4^t \int \int \xi^{p+\lambda\beta} \, dx \, dt.$$

Moser iteration procedure results in

$$\left( \int \int \xi v^{p+2\alpha+2\lambda} \, dx \, dt \right)^{1/2} \leq \frac{C}{R^{p+2}} \int \int v^p \, dx \, dt.$$

Thus

$$\text{ess sup} \xi v \leq \frac{C}{R^{p+2}} \left( \int \int |\nabla u|^p \, dx \, dt \right)^{1/2}. \quad \text{(2.14)}$$

We have now completed the proof under the assumption that $u_s, u_t, u_{s_t}, \nabla u_{s_t}$, are in suitable $L^p$ spaces so that the formal calculation can be justified. For the general case, we can average with respect to $t$ and finite-difference with respect to $x$ (refer to the explanation on p. 95 in [3]). The theorem has been proved.

As a by-product we have the following theorem:

**Theorem 2.2.** For any $s > 0, T > 0$
\[
\iint_{\Omega} |\nabla u|^{p-2} |D^2 u| |dx|^{2} dt \leq C_{**} \quad \text{if } p > 2, \tag{2.15}
\]

and
\[
\iint_{\Omega} |\nabla u|^{2p-2} |D^2 u| |dx|^{2} dt \leq C_{**} \quad \text{if } \frac{3}{2} < p < 2, \tag{2.16}
\]

for any solution \( u \) of \((2.1)\) with \( \|u\|_{p} \leq \Gamma \), where \( C_{**} \) is a positive constant depending only on \( p, N, T, s \) and \( \Gamma \).

**Proof** If \( p > 2 \), \((2.5)\) with \( \alpha = 0 \) gives \((2.15)\).

If \( 2 > p > \frac{3}{2} \), combining \((2.5)\) with \((2.6)\) and setting \( \alpha = \frac{p-2}{2} \) we can obtain
\[
\frac{1}{\mathcal{P}} \sup_{r-R < t < r^{*}} \iint_{B(R)} |\nabla u|^2 \|dx\| \|dt\| + \left( 2p-3 \right) \left( \iint_{\Omega} |\nabla u|^{2p-2} |D^2 u| |dx|^{2} dt \right)^{\frac{1}{p}} \left( \iint_{\Omega} |\nabla u|^{2p-2} |D^2 u| |dx|^{2} dt \right)^{\frac{1}{2}} \leq C \left( \iint_{\Omega} |\nabla u|^{2p-2} |\nabla u|^{2p-2} |D^2 u| |dx|^{2} dt \right)^{\frac{1}{p}} \left( \iint_{\Omega} |\nabla u|^{2p-2} |D^2 u| |dx|^{2} dt \right)^{\frac{1}{2}}.
\]

It is easy to get \((2.16)\) from this.

\section*{§ 3. Local Properties of \( \nabla u \)}

Suppose that \( u(x, t) \) is a solution of \((2.1)\) in some cylindrical domain
\[
Q_{(R)} = \{(x, t) \mid |x| < R, -\frac{R^2}{\mu^{p-2}} < t \leq 0 \}, \tag{3.1}
\]

where \( 0 < R < 1 \) and \( \mu > 0 \) are some constants.

Set
\[
M_{T, \mu}^{j} (R) = \sup_{Q_{(R)}} (\pm u_{j}), \quad M_{\mu} (R) = \max_{1 < i < N} \sup_{Q_{(R)}} |u_{i}|.
\]

**Lemma 3.1.** Let \( \mu \) satisfy
\[
2M_{T, \mu}^{j} (R) \geq \mu \geq M_{\mu} (R). \tag{3.2}
\]

Then there exists \( \varepsilon_{0} \), depending only on \( p \) and \( N \), such that
\[
\iint_{Q_{(R)}} [M_{T, \mu}^{j} (R)]^{a} \left| u_{j} ight|^{a-1} u_{j} \left| u_{j} \right|^{2} dx dt \leq \varepsilon_{0} [M_{T, \mu}^{j} (R)]^{2a} \tag{3.3}
\]

implies
\[
\inf_{Q_{(R)}} u_{j} \geq \frac{M_{T, \mu}^{j} (R)}{2^{a}}, \tag{3.4}
\]

where \( a = \min \{1, p-1\} \) and
\[
\iint_{Q_{(R)}} f \|dx\| \|dt\| = \frac{1}{\text{meas} \ Q_{(R)}} \iint_{Q_{(R)}} f \|dx\| \|dt\|.
\]

**Proof** Assume for the moment \( u \) is a smooth solution of \((2.1)\). Upon differentiating with respect to \( x_{1} \), we derive the equation
\[
\frac{\partial u_{x_{1}}}{\partial t} - (a_{ij} |\nabla u|^{p-2} u_{x_{1}})_{x_{i}} = 0, \tag{3.5}
\]
where
\[ \alpha = \delta_{u} + \frac{(p-2)\|u_{0}\|_{p}}{\|u\|_{p}} \quad \text{if} \quad |\nabla u| \neq 0. \quad (3.6) \]

Obviously we have
\[ \min\{p-1, 1\}|\xi|^{2} < u_{i} \xi_{i} \xi_{j} < \max\{p-1, 1\}|\xi|^{2}, \quad \forall \xi \in \mathbb{R}^{N}. \quad (3.7) \]

Set
\[ v = |u_{a}|^{a-1}u_{a}, \quad (3.8) \]

which means \( u_{a} = |v|^{\frac{1}{a}} \) \text{sign} \( v \). The function \( v \) satisfies in the weak sense the equation
\[ \frac{1}{a}|v|^{\frac{1}{a}} \frac{\partial v}{\partial t} - \left( \frac{1}{a} \alpha_{u} |\nabla u|^{p-2} |v|^{\frac{1-a}{a}} v_{i} \right)_{i} = 0. \quad (3.9) \]

Let \( \zeta(x) \) be the usual cut-off function in \( \mathbb{Q}_{u}(R) \) and set \( M_{1} = M_{1}^{+}(R) \).

Applying the test function \( \zeta^{2}(v - k)^{-} = \zeta^{2} \max\{k - v, 0\} \) to (3.9) for \( \frac{M_{1}^{a}}{2} < k < M_{1}^{a} \) and setting
\[ \zeta_{k}(t) = \int_{0}^{t} |k - s|^{\frac{1-a}{a}} s ds \quad (3.10) \]

we have
\[ \iint_{Q_{u}(\mathbb{R})} \zeta^{2} \frac{\partial \zeta_{k}(v - k)^{-}}{\partial t} dx \, dt \, + \, \iint_{Q_{u}(\mathbb{R})} \zeta^{2} \alpha_{u} |\nabla u|^{p-2} |v|^{\frac{1-a}{a}} v_{i} v_{i} \, dx \, dt \]
\[ = 2 \iint_{Q_{u}(\mathbb{R})} \zeta \alpha_{u} |\nabla u|^{p-2} |v|^{\frac{1-a}{a}} (v - k)^{-} \zeta_{k} v_{i} v_{i} \, dx \, dt. \quad (3.11) \]

Now define
\[ \phi_{k}(s) = \begin{cases} 0 & \text{if } s > k \\ k - s & \text{if } k \leq s < k - \frac{M_{1}^{a}}{4} \\ \frac{M_{1}^{a}}{4} & \text{if } s < k - \frac{M_{1}^{a}}{4}. \end{cases} \quad (3.12) \]

Noting that \( \frac{M_{1}^{a}}{2} < k < M_{1}^{a} \) and (3.2) we can get
\[ \phi_{k}(v) \leq (v - k)^{-} \leq C \phi_{k}(v). \]

After simple computation we find
\[ \iint_{Q_{u}(\mathbb{R})} \zeta^{2} \frac{\partial \phi_{k}(v)_{i}}{\partial t} dx \, dt \, + \, \min\{p-1, 1\} \iint_{Q_{u}(\mathbb{R})} \zeta^{2} |\nabla u|^{p-2} |v|^{\frac{1-a}{a}} |\nabla \phi_{k}(v)|^{2} dx \, dt \]
\[ \leq C \iint_{Q_{u}(\mathbb{R})} |\nabla u|^{p-2} |v|^{\frac{1-a}{a}} [(v - k)^{-}]^{2} |\nabla \zeta|^{2} dx \, dt, \]

and then
\[ \text{ess sup} \int_{D(\mathbb{R})} \zeta^{2} \phi_{k}(v)_{i} dx + \iint_{Q_{u}(\mathbb{R})} \zeta^{2} |\nabla u|^{p-2} |v|^{\frac{1-a}{a}} |\nabla \phi_{k}(v)|^{2} dx \, dt \]
\[ \leq C \iint_{Q_{u}(\mathbb{R})} |\nabla u|^{p-2} |v|^{\frac{1-a}{a}} \phi_{k}(v)_{i} |\nabla \zeta|^{2} dx \, dt + \iint_{Q_{u}(\mathbb{R})} \zeta^{2} \phi_{k}(v)_{i} dx \, dt. \quad (3.13) \]
Obviously we have, noting that \( \frac{M_1^2}{2} \leq k \leq M_2^2 \) and (3.2),

\[
\frac{1}{C} \mu^{1-a} \phi_b^2(v) \leq \chi_b \left( \phi_b(v) \right) \leq C \mu^{1-a} \phi_b^2(v),
\]

\[
| \nabla u |^{p-2} |v|^{1-a} \leq N^{\frac{n-2}{2}} \mu^{p-a-1},
\]

and, if \( k \geq v \geq k - \frac{M_2^2}{4} \),

\[
| \nabla u |^{p-2} |v|^{1-a} \geq \frac{1}{C} \mu^{p-a-1}.
\]

Thus it follows from (3.11) that

\[
\text{ess sup} \int_{B(R)} \xi^2 \phi_b^2(v) \ dx + \mu^{p-2} \iint_{\Omega_n(R)} | \nabla \left( \xi \phi_b(v) \right) |^2 \ dx \ dt 
\]

\[
\leq C \iint_{\Omega_n(R)} \phi_b^2(v) [ \mu^{p-2} | \nabla \xi |^2 + | \xi_t | ] \ dx \ dt.
\]

Let \( \tau = t \mu^{p-2} \) and

\[
\mathcal{Q}(R) = \{(x, \tau) \mid |x| < R, -R^2 < \tau < 0 \}.
\]

Then we obtain

\[
\text{ess sup} \int_{B(R)} \xi^2 \phi_b^2(v) \ dx + \mu^{p-2} \iint_{\mathcal{Q}(R)} | \nabla \left( \xi \phi_b(v) \right) |^2 \ dx \ dv \tau
\]

\[
\leq C \iint_{\mathcal{Q}(R)} \phi_b^2(v) [ \mu^{|x|^2} (x) | \nabla \xi |^2 + | \xi_t | ] \ dx \ dv \tau.
\]

According to Ladyzenskaja et al. ([4], p. 75) again, we find

\[
\left[ \iint_{\mathcal{Q}(R)} \left( \xi \phi_b(v) \right)^{\frac{2(n+2)}{n}} \ dx \ dv \tau \right]^{\frac{n}{n+2}} \leq C \iint_{\mathcal{Q}(R)} \phi_b^2(v) [ \mu^{|x|^2} (x) | \nabla \xi |^2 + | \xi_t | ] \ dx \ dv \tau.
\]

Back to the original time variable \( t \), we have

\[
\left[ \iint_{\mathcal{Q}(R)} \left( \xi \phi_b(v) \right)^{\frac{2(n+2)}{n}} \ dx \ dt \right]^{\frac{n}{n+2}} \leq C \mu^{\frac{2(p-2)}{n+2}} \iint_{\mathcal{Q}(R)} \phi_b^2(v) [ \mu^{|x|^2} (x) | \nabla \xi |^2 + | \xi_t | ] \ dx \ dt.
\]

Set

\[
k_l = M_l \left( \frac{1}{2} + \frac{1}{2^{l+1}} \right),
\]

\[
R_l = R \left( \frac{1}{2} + \frac{1}{2^{l+1}} \right),
\]

and let \( \xi_l(x, t) \) be the cut-off function in \( Q_n(R_l) \) satisfying

\[
\xi_l(x, t) = 1 \quad \text{in} \ Q_n(R_{l+1}),
\]

\[
0 \leq \xi_l(x, t) \leq 1,
\]

\[
| \nabla \xi_l |^2 \leq \frac{C_4 l}{R^2}, \quad | \xi_l_t | \leq \frac{C_4 l \mu^{p-2}}{R^2}.
\]

Applying (3.14) in \( Q_n(R_l) \) we get

\[
\left[ \iint_{\mathcal{Q}(R_l)} \left( \xi \phi_b(v) \right)^{\frac{2(n+2)}{n}} \ dx \ dt \right]^{\frac{n}{n+2}} \leq \frac{C_4 l \mu^{\frac{2(p-2)}{n+2}}}{R^2} \iint_{\mathcal{Q}(R_l)} \phi_b^2(v) \ dx \ dt.
\]

(3.15)
Now define

\[ J_i = \iint_{Q_i(R)} \phi_{i+1}^2(v) \, dx \, dt. \]

In view of Hölder inequality it follows that

\[ J_{i+1} = \iint_{Q_i(R)} \phi_{i+1}^2(v) \, dx \, dt \leq \left[ \iint_{Q_i(R)} [\phi_{i+1}(v)]^{2(N+2)/N} \, dx \, dt \right]^{N/2} \left[ \text{meas } Q_i(R_{i+1}) \cap \{\phi_{i+1}(v) > 0\} \right]^{2/(N+2)}. \tag{3.16} \]

Notice that

\[ J_i \geq \frac{1}{C} \iint_{Q_i(R)} [(v - k_t)^2] \, dx \, dt \geq \frac{1}{C} (k_t - k_{i+1})^2 \text{meas } Q_i(R_{i+1}) \cap \{v < k_{i+1}\} \]

\[ \geq \frac{M_1^2 a_0^2}{C^2} \text{meas } Q_i(R_{i+1}) \cap \{\phi_{i+1}(v) > 0\}. \tag{3.17} \]

The inequality (3.15), (3.16) and (3.17) results in

\[ J_{i+1} \leq C 16 \mu^{2(N+2)/N+2} J_i^{1+2/(N+2)}. \]

Set

\[ Y_i = J_i \mu^{p-2}/M_1^2 a_0 R^{N+2}. \]

The previous inequality can be written in the following

\[ Y_{i+1} \leq C 16 Y_i^{1+2/(N+2)}. \]

According to [4, Lemma II. 5.6, p. 95] therefore

\[ Y_i \rightarrow 0 \text{ if } l \rightarrow \infty, \tag{3.18} \]

provided

\[ Y_0 < \delta_0, \]

which means (3.3) for some \( \delta_0 \). The fact (3.18) implies

\[ \text{ess inf } v > \frac{M_1^2}{2}, \]

and (3.4) follows at once.

This proves the lemma under the additional assumption that \( u \) is a smooth solution of (2.1). In the general case we recall Theorem 2.2 and prove by routine arguments that \( v \) is a weak solution of (3.9): note \(|\nabla u|^{p-2}|\nabla u|^{1-\eta/2} |\nabla u|^{2} \in L(Q_i(R))\).

The rest of the proof is the same.

**Lemma 3.2.** Suppose that \( \mu \) satisfies

\[ 2M^{1+\eta}_\mu(R) > \mu > M^\mu(R). \]

Then for any \( \delta_0 > 0 \) there exist constants \( 0 < \lambda, \beta < 1 \), depending only on \( p, N \) and \( \delta_0 \), such that we have

\[ \text{meas } \{(x, t) \in Q_i(R) | u_x < (1 - \beta) M^{1+\eta}_\mu(R) \} > \lambda \text{meas } Q_i(R), \tag{3.19} \]

if the inequality
\[
\iiint_{Q_{\alpha}^c} [(M_{1,\mu}^+ R)^{\alpha} - |u_{\alpha} - u_{\alpha}|^2] \, dx \, dt \leq \varepsilon \left( M_{1,\mu}^+ R \right)^{\alpha} \tag{3.20}
\]
fails, where \(\alpha = \min\{p - 1, 1\}\).

**Proof** Suppose that (3.19) fails (for \(\beta, \lambda\) as selected below) and set as before \(M_1 = M_{1,\mu}^+ R\). Then

\[
\iiint_{Q_{\alpha}^c} (M_1^+ - |u_{\alpha} - u_{\alpha}|^2) \, dx \, dt = \iiint_{Q_{\alpha}^c \cap \{(u_{\alpha} < (1-\beta) M_1^+ R)\}} + \iiint_{Q_{\alpha}^c \cap \{(u_{\alpha} > (1-\beta) M_1^+ R)\}} \leq \lambda \text{meas } Q_{\alpha}^c R + (\beta M_1^2)^\alpha \text{meas } Q_{\alpha} R \quad (\beta < \frac{1}{2})
\]

\[
\leq [\lambda (1+2^\alpha)^\alpha + \beta^{2\alpha} M_1^{2\alpha}] \text{meas } Q_{\alpha} R .
\]

Select \(\lambda\) and \(\beta\) such that

\[(1+2^\alpha)^\alpha + \beta^{2\alpha} \leq \varepsilon_0, \quad \beta < \frac{1}{2} .
\]

The lemma has been proved.

**Lemma 3.3.** Suppose that

\[2M_{1,\mu}^+ R \geq \mu \geq M_{\mu} R ,
\]
and there exist constants \(0 < \lambda, \beta < 1\) such that

\[\text{meas } \{(x, t) \in Q_{\mu} R \mid u_{\mu} (x, t) < (1-\beta) M_{1,\mu}^+ R \} \geq \lambda \text{meas } Q_{\mu} R . \tag{3.21}
\]

Then there exist constants \(0 < \delta, \gamma < 1\), depending only on \(N, p, \lambda\) and \(\beta\), such that

\[M_{1,\mu}^+ (8R) \leq \gamma M_{1,\mu}^+ (R) . \tag{3.22}
\]

**Proof** For any \(s > 0\), define

\[\psi(s) = \begin{cases} (s^2 + s^\alpha)^{\frac{1}{2}} - s, & \text{if } s \geq 0, \\ 0, & \text{if } s < 0, \end{cases}
\]

and set

\[k = (1-\beta) M_{1,\mu}^+ R, \quad w = (u_{\mu} - k)^+, \quad w_{\mu} (x, t) = \psi(u_{\mu} - k).
\]

It is easy to get, for any \(\phi \in C_0^\infty (Q_{\mu} R)\) and \(\phi \geq 0\),

\[\iiint_{Q_{\mu}^c} [w_{\phi} \psi - a_{ij} |\nabla u|^{p-2} w_{\phi} \partial w_{\phi}] \, dx \, dt \geq 0 .
\]

Let \(s \to 0\), we can get, for any \(\phi \in C_0^\infty (Q_{\mu} R)\) and \(\phi \geq 0\),

\[\iiint_{Q_{\mu}^c} [\tilde{w}\phi - \tilde{\tilde{a}}_{ij} w_{\phi}] \, dx \, dt \geq 0 ,
\]

where

\[\tilde{\tilde{a}}_{ij} = \begin{cases} a_{ij} |\nabla u|^{p-2}, & \text{if } u_{\mu} \geq k, \\ w^{p-2} \delta_{ij}, & \text{if } u_{\mu} < k. \end{cases}
\]

This means that \(w(x, t)\) satisfies in the weak sense

\[w_{\mu} - (\tilde{a}_{ij} w_{\phi})_{x_i} \leq 0 \quad \text{in } Q_{\mu} R .
\]
Let $\tau = \mu^{p-2}$. Then $w(x, \tau)$ satisfies in the weak sense

$$w_w - \left(\frac{\tilde{a}_{ij}}{\mu^{p-2}} w_{x_i}\right) \leq 0 \quad \text{in} \ Q(R),$$

where $Q(R) = \{(x, \tau) \mid |x| < R, -R^2 < \tau < 0\}$. It is easy to observe

$$\frac{1}{C} |\xi|^2 \leq \tilde{a}_{ij} \xi_i \xi_j \leq C |\xi|^2, \forall \xi \in \mathbb{R}^N$$

and $(x, \tau) \in \bar{Q}(R)$,

where $C$ depends only on $N, p$ and $\beta$.

Define

$$\tilde{a}_{ij} = \frac{a_{ij}}{\mu^{p-2}}, \quad v = \frac{\text{ess sup} w - w}{\text{ess sup} w}.$$

The assumption (3.21) implies

$$\text{meas} \{(x, \tau) \in \bar{Q}(R) \mid v(x, \tau) \geq 1\} \geq \lambda \text{meas} \bar{Q}(R).$$

The function $v(x, \tau)$ satisfies in the weak sense

$$v_w - (\tilde{a}_{ij} v_{x_i}) x_i \geq 0.$$

By an estimate of Kruzhkov (see the appendix in [1]) there exist constants $0 < \delta, \gamma < 1$ such that

$$\text{ess inf} v \geq 1 - \gamma > 0,$$

that is

$$\text{ess sup} w \leq \gamma \text{ess sup} w.$$

Back to the original time variable $t$, we have

$$M_t^1(\delta R) \leq \gamma M_t^1(R)$$

as claimed.

§ 4. Interior Hölder Continuity of $\nabla u$

Now we determine $s_0$ by Lemma 3.1, and then constants $\lambda, \beta$ by $s_0$ and Lemma 3.2, and finally $\delta, \gamma$ by $\lambda, \beta$ and Lemma 3.3. In this case, $\delta$ and $\lambda$ depend only on $N$ and $p$.

Select a constant $s$: $1 < s < 2$ if $p > 2$ and $2 < s < 3$ if $1 < p < 2$, such that

$$\delta \frac{2^{(2-s)/s} - 1}{(s-2)/s} \geq \max \left\{ \frac{1}{2}, \gamma \right\}. \quad (4.1)$$

It suffices to choose $s$ close to 2. Let $\Omega_p^c \subset \subset \Omega_p$ and suppose

$$||\nabla u||_{L^p} \leq \bar{M}_p,$$

and define

$$M_\delta = \bar{M}_p \delta \frac{2^{(2-s)/s} - 1}{(s-2)/s}.$$

Fix the point $P_0(x^0, t^0) \in \Omega_p^c$ and set

$$Q_\delta(P_0, R) = \left\{(x, t) \mid |x - x^0| < R, \frac{R^2}{\mu^{s-2}} < t < t^0 \right\}. \quad (4.8)$$

The constant $0 < R_0 < 1$ is selected such that

$$Q_{2M_\delta}(P_0, R_0) \subset \Omega_p^c. \quad (4.4)$$

Set

$$Q_{\lambda M_\delta}(P_0, R_0) \subset \Omega_p^c.$$
\( \hat{Q}(P_0, R) = \left\{ (x, t) \mid |x - x_0| < R, t^0 - \frac{R^2 R_0^{2-\frac{\gamma}{\alpha}}}{(2M_0)^{\alpha-2}} < t < t^0 \right\} \) \quad (4.5)

and

\[
\begin{align*}
M_i^+(R) &= \text{ess sup}_{(P_0, R)} (\pm u_{x_i}) \quad (i = 1, 2, \ldots, N), \\
M_i^-(R) &= \text{ess sup}_{(P_0, R)} |u_{x_i}|, \\
M_i(R) &= \max_{1 \leq i \leq N} \text{ess sup}_{(P_0, R)} |u_{x_i}|, \\
\text{osc}_{(P_0, R)} u_{x_i} &= \text{ess sup}_{(P_0, R)} u_{x_i} - \text{ess inf}_{(P_0, R)} u_{x_i} = M_i^+(R) + M_i^-(R).
\end{align*}
\]

**Theorem 4.1.** Let \( u \) be a solution of the equation (2.1). Then there exist constants \( 0 < p < 1 \) and \( C > 0 \), depending only on \( N \) and \( p \), such that

\[
\begin{align*}
\text{osc}_{(P_0, R)} u_{x_i} &< C M_0 \left( \frac{R}{R_0} \right)^p \quad \text{for } 0 < R < R_0, \quad i = 1, 2, \ldots, N, \\
\text{osc}_{Q_a(P_0, R)} u_{x_i} &< C M_0 \left( \frac{R}{R_0} \right)^p \quad \text{for } 0 < R < R_0, \quad i = 1, 2, \ldots, N.
\end{align*}
\]

\( \delta \frac{3}{2} < p < 2 \) and

\( \delta \frac{p}{2} < p < 2 \).

**Proof.** Define

\[
R_a = \sup \left\{ R \in [0, R_0] \mid \exists 1 < j < N, \theta \in \{\pm, -\} \mid M_j^+(R) > 2M_0 \left( \frac{R}{R_0} \right)^{\frac{2-\delta}{p-2}} \right\}. \quad (4.9)
\]

Then we can suppose \( R > 0 \), since otherwise the theorem has been proved. In addition we have \( R_a \leq \delta \frac{2}{3} R_0 \) by the definition of \( M_0 \) and \( \hat{M}_0 \).

Thus, there must exist \( R_a \):

\[
\delta \frac{2}{3} R_0 < R_a < R < R_0
\]

such that we have

\[
|M_j^+(R_a)| < 2M_0 \left( \frac{R}{R_0} \right)^{\frac{2-\delta}{p-2}} \quad \text{for } j = 1, 2, \ldots, N, \quad (4.11)
\]

and there exist \( \delta_0 \) and \( \theta \), say \( \delta_0 = 1, \theta = + \), such that

\[
M_j^+ (\delta^j \frac{2}{3} R_0) > 2M_0 \left( \frac{\delta^j \frac{2}{3} R_0}{R_0} \right)^{\frac{2-\delta}{p-2}}. \quad (4.12)
\]

Set

\[
\mu = 2M_0 \left( \frac{R_2}{R_0} \right)^{\frac{p-2}{p-\delta}}. \quad (4.13)
\]

At first we prove that

\[
\iint_{Q_a(P_0, R_2)} [(M_1^+(R_2))^a - |u_{x_1} - u_{x_2}|^a] \, dx \, dt < \varepsilon_0((M_1^+ R_2))^2a \quad (4.14)
\]

where \( Q_a(P_0, R_2) \) is defined in (4.3).

For \( R < R_2 \), define

\[
M_1^+(R) = \text{ess sup}_{Q_a(P_0, R)} u_{x_1}, \quad M_2^+(R) = \max_{1 \leq i \leq N} |u_{x_i}|. \quad (4.15)
\]

Notice that \( Q_a(P_0, R_2) = \hat{Q}(P_0, R_2) \) and \( M_1^+(R_2) = M_1^+(R) \).
By (4.1) and (4.13), (4.12) implies
\[ M^{+}_{1,m}(R_{2}) > M^{+}_{1}(\delta^{3}R_{2}) > \max \left\{ \frac{1}{2}, \mu \right\} \mu. \] (4.16)

Then we have, noting (4.11), (4.13) and (4.16),
\[ 2M^{+}_{1,m}(R_{2}) > \mu > M_{m}(R_{2}). \] (4.17)

If (4.14) fails, it follows by Lemma 3.2 that
\[ \text{meas}\{(x, t) \in Q_{\alpha}(R_{2}) \mid u_{\alpha} \leq (1 - \beta) M^{+}_{1,m}(R_{2})\} > \lambda \text{meas} Q_{\alpha}(R_{2}). \] (4.18)

By Lemma 3.3, (4.17) and (4.18) result in
\[ M^{+}_{1,m}(\delta R_{2}) < 2^{2} M_{m}(R_{2}) \]

Noting that \( Q_{\alpha}(P_{0}, \delta R_{2}) = \dot{Q}(P_{0}, \delta^{3}R_{2}) \), we get
\[ M^{+}_{1}(\delta^{3}R_{2}) = M^{+}_{1,m}(\delta R_{2}) < 2^{2} M_{m}(R_{2}) \]
which contradicts (4.16). (4.14) has been proved.

By Lemma 3.1, (4.14) and (4.17) implies
\[ \text{essinf}_{Q_{\alpha}(P_{0}, \delta R_{2})} u_{\alpha} > \frac{M^{+}_{1}(R_{2})}{2^{2}} \left( \frac{1}{2} \right) \mu. \] (4.19)

We know that \( u_{\alpha} \) satisfies in the weak sense
\[ \frac{\partial u_{\alpha}}{\partial t} - (a_{ij} |\nabla u|^{p-2} u)_{t,i} = 0, \]
where \( a_{ij} \) are defined in (3.6).

Set
\[ \xi = x - x^{0}, \quad \tau = \mu^{p-2}(t - t^{0}), \]
\[ Q'(R) = \{ (\xi, \tau) \mid |\xi| < R^{2}, -R^{2} < \tau < 0 \}, \]
\[ v(\xi, \tau) = u_{\alpha}(x, t). \]

Then \( v \) satisfies in the weak sense
\[ \frac{\partial v}{\partial \tau} - \left( a_{ij} \frac{|\nabla u|^{p-2}}{\mu^{p-2}} v_{t,i} \right)_{t,i} = 0. \]

In view of (4.19), it follows that
\[ \frac{1}{C} |\eta|^{p} < a_{ij} \frac{|\nabla u|^{p-2}}{\mu^{p-2}} |\eta_{ij}|^{p} \leq C |\eta|^{p}, \quad \forall \eta \in \mathbb{R}^{N}, (\xi, \tau) \in Q'(\frac{R_{2}}{2}). \]

Applying Hölder interior estimates for non-degenerate equations, we have
\[ \text{osc} v < C \left( \frac{R}{R_{2}} \right)^{\frac{\beta}{2}} \text{osc} v \text{ for } 0 < R < \frac{R_{2}}{4}, \]
where \( C > 0 \) and \( 0 < \beta < 1 \) depend only on \( N \) and \( p \).

Back to the variables \((x, t)\), we obtain
\[ \text{osc}_{Q_{\alpha}(P_{0}, R_{2})} u_{\alpha} < C \left( \frac{R}{R_{2}} \right)^{\frac{\beta}{2}} \text{osc} u_{\alpha} \text{ for } 0 < R < \frac{R_{2}}{4}, \quad k = 1, 2, \ldots, N. \] (4.20)

If \( R \geq R_{2} \), then by the definition of \( R_{2} \)
\[ \text{osc}_{Q_{\alpha}(P_{0}, R_{2})} u_{\alpha} \leq |M^{+}_{1}(R)| + |M^{-}_{1}(R)| < 4M_{0} \left( \frac{R}{R_{0}} \right)^{\frac{2-\beta}{2}}. \] (4.21)
If \( \frac{R_2}{4} < R \leq R_2 \), then

\[
\text{osc}_Q u_n \leq \text{osc}_{Q(R, R_2)} u_n \leq 4M_0 \left( \frac{4R}{R_0} \right)^{2-s}.
\]  

(4.22)

If \( 0 < R < \frac{R_2}{4} \), then it follows from (4.20) and (4.22) that

\[
\text{osc}_{Q_R} u_n \leq C \left( \frac{R}{R_2} \right)^2 4M_0 \left( \frac{4R}{R_0} \right)^{2-s}.
\]

Set \( \rho = \min \left\{ \beta, \frac{2-s}{p-2} \right\} \), we obtain

\[
\text{osc}_{Q_R} u_n \leq \text{osc}_{Q_{R_2}} \left( \frac{R}{R_2} \right)^\rho \left( \frac{R_2}{R_0} \right)^\rho = \text{osc}_{Q_{R_0}} \left( \frac{R}{R_0} \right)^\rho.
\]

(4.23)

If \( 1 < p < 2 \), then \( Q_{R_0}(P_0, R) \supset Q_{P_0}(P_0, R) \) for \( 0 < R < \frac{R_2}{4} \), and if \( p > 2 \) then

\( Q_{P_0}(P_0, R) \supset Q_{R_0}(P_0, R) \) for \( 0 < R \leq \frac{R_2}{4} \) and \( Q_{Q_{R_0}}(P_0, R) \supset Q_{Q_{R_0}}(P_0, R) \) for \( 0 < R \leq \frac{R_2}{4} \).

Thus (4.21), (4.22) and (4.23) imply what we want.

References


