STATIONARY SOLUTIONS OF THE RELATIVISTIC
VLASOV-MAXWELL SYSTEM OF PLASMA PHYSICS

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Abstract

The authors consider the stationary relativistic coupled system consisting of Vlasov's equation for the distribution function of charged particles and Maxwell's equations for the electric and magnetic fields of a plasma. With different tools of nonlinear functional analysis the existence of solutions is proved, in which, according to different geometries and symmetries, the distribution function depends on one, two or three independent integrals of the motion.

Keywords Relativistic Vlasov-Maxwell system, Stationary solutions, Systems of semilinear elliptic equations

1991 MR Subject Classification 82A45, 35J65

§1. Introduction

The present paper is part of a mathematical description of a collisionless plasma considered as a collection of many fast moving charged particles whose collisions are neglected and which interact only by their charges. The basic equations for the time development of the species of electrons consist of the following system of partial differential equations which is now called the Relativistic Vlasov-Maxwell System (RVMS)

\[
\begin{align*}
\partial_t f + \hat{v} \partial_x f - q \left( E + \frac{1}{c} \cdot \hat{v} \times B \right) &= 0, \\
\frac{1}{c} \partial_t E - \text{curl} B &= - \frac{4\pi}{c} j, \\
\frac{1}{c} \partial_t B + \text{curl} E &= 0, \\
\text{div} E &= 4\pi \rho, \\
\text{div} B &= 0.
\end{align*}
\]

Here \( f = f(t, x, v) \geq 0 \) denotes the distribution function of the electrons depending upon the time \( t \geq 0 \), the space coordinate \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) and the momentum \( v = (v_1, v_2, v_3) \in \mathbb{R}^3 \), \( E = E(t, x) \) and \( B = B(t, x) \) are the electric and magnetic fields and \( \hat{v} := \frac{v}{\sqrt{m^2 + v^2/c^2}} \) is the relativistic speed of a particle, \( q \) and \( m \) denote the charge and mass of a particle, \( c \) is the speed of light and \( \rho = \rho(t, x) \) and \( j = j(t, x) \) are the local charge and current densities.
These two quantities are related with \( f \) through
\[
\rho(x) := q \int_{\mathbb{R}^3} f(x, v) \, dv,
\]
\[
j(x) := q \int_{\mathbb{R}^3} \dot{v} f(x, v) \, dv.
\]

The associated initial value problem with a prescribed distribution \( f_0 \)
\[
f(0, x, v) = f_0(x, v)
\]
has received much attention in recent years. In the classical setting, R.T. Glassey and W.A. Strauss have shown that a local solution is actually global if there exists an a priori bound for the support of \( f(t, x, v) \) in the momentum variable on the interval of its existence\(^6\), or more generally, if the kinetic energy density
\[
\int_{\mathbb{R}^3} (m^2 + v^2)^{1/2} f(t, x, v) \, dv
\]
stays bounded\(^{10}\). These results were used by R.T. Glassey, J. Schaeffer and W.A. Strauss to obtain the existence of global classical solutions for small initial data\(^9\) or nearly neutral data\(^{11}\). G. Rein considered a class of global solutions with a certain asymptotic behavior of the resulting solutions\(^{17}\). Global existence of classical solutions for general initial data is still an open problem. Global existence of distributional solutions were proven by P.L. Lions and R. Di Perna\(^{12}\). In contrast to the classical situation, there is no uniqueness result for these weak solutions.

The present paper is concerned with the existence of stationary solutions of the RVMS in several geometric configurations arising from known solutions of Vlasov's equation (1.1), that is, from three known integrals of the associated system of ordinary differential equations
\[
x = \dot{v},
\]
\[
\dot{v} = q(E + \frac{1}{c} \cdot \dot{\varphi} \times B).
\]

Let us denote by \( \Phi = \Phi(x) \) and \( A = A(x) \) the scalar and vector potential associated with Maxwell's equations (1.2)-(1.5). A first integral is the energy density
\[
\mathcal{E}(x, v) := \Phi(x) + \sqrt{1 + v^2}.
\]

We get a second integral if we assume that \( \Phi \) and \( A \) are cylindrically symmetric, that is, they only depend upon \( r(x) := \sqrt{x_1^2 + x_2^2} \) and \( z := x_3 \) (but not upon \( \vartheta \) in cylindrical coordinates \( r, \vartheta, z \), namely
\[
F(r, z, v) := r(\dot{v}_\vartheta + A_\vartheta(r, z)).
\]

(Indices \( r, \vartheta, z \) denote the components of a vector in \( \mathbb{R}^3 \) in the local coordinate system \((e_r(x), e_\vartheta(x), e_z(x))\).) If we assume that \( \Phi \) and \( A \) are translational invariant, such that they do not depend upon \( z \), then the quantity
\[
P(r, \vartheta, v) := v_z + A_z(r, \vartheta)
\]
is a third integral. These integrals are well known in plasma physics. In a recent note P. Degond has set up the form of systems of equations whose solution might lead to the con-
struction of stationary solutions of the RVMS\[^{[5]}\]. In fact, the determination of a distribution \( f \) as a solution of Vlasov's equation in one of the following forms

\begin{align*}
\text{Case 1: } f &= \varphi(\mathcal{E}), \\
\text{Case 2: } f &= \varphi(\mathcal{E}, F), \\
\text{Case 3: } f &= \varphi(\mathcal{E}, F, P),
\end{align*}

requires the solution of one semilinear elliptic equation in Case 1), of a system of two semilinear elliptic equations (one containing a singular term for \( r = 0 \)) in Case 2) and a system of three ordinary differential equations of second order (one again containing a singular term) in Case 3), subject to suitable boundary conditions.

It is the purpose of the present paper to prove the existence of solutions for the resulting equations with the boundary conditions of a perfect conductor and thus to get three essentially different families of the stationary RVMS.

The existing literature is not yet very rich in the topic addressed here. However, if we formally let \( B = 0 \) or let \( c \to \infty \) then we obtain the well known (relativistic) Vlasov-Poisson system of equations denoted by (R)VPS. Stationary solutions of the RVPS have been constructed in \([4]\) and for the classical VPS in \([2]\) and \([3]\). These articles have influenced the present investigation. For two species of particles and for given distribution functions depending only upon the energy, G. Rein has recently proven the existence and uniqueness of stationary solutions of the RVMS by variational methods\[^{[18]}\]. The articles of J. Dolbeault\[^{[6]}\], of F. Poupaud\[^{[16]}\] and of the Russian School at Irkutsk\[^{[13,14]}\] contribute further to an expanding theory.

\section*{§2. Formulation of the Problem}

Let \( \Omega \subset \mathbb{R}^3 \) be a domain with boundary \( \partial \Omega \in C^1 \). For the sake of simplicity we let \( q = m = c = 1 \) and consider the following system of equations:

\begin{align}
\dot{\varrho} \partial_v f - (E(x) + \dot{v} \times B(x))\partial_v f &= 0, \\
\mathrm{curl} \, B(x) &= 4\pi j(x), \\
\mathrm{curl} \, E(x) &= 0, \\
\mathrm{div} \, E(x) &= 4\pi \rho(x), \\
\mathrm{div} \, B(x) &= 0, \quad x \in \Omega, v \in \mathbb{R}^3
\end{align}

\begin{align}
\rho(x) := \int_{\mathbb{R}^3} f(x, v) \, dv,
\end{align}

\begin{align}
j(x) := \int_{\mathbb{R}^3} \dot{v} \, f(x, v) \, dv, \quad x \in \Omega.
\end{align}

We shall impose the boundary conditions of an ideal conductor, that is,

\begin{align}
E(x) \times \nu(x) &= 0, \\
\langle B(x), \nu(x) \rangle &= 0, \quad x \in \partial \Omega,
\end{align}

where \( \nu(x) \) is the outer normal in \( x \in \partial \Omega \). A triple of functions \((f, E, B)\) with \( f \in C(\overline{\Omega} \times \mathbb{R}^3) \cap C^1(\Omega \times \mathbb{R}^3), f \geq 0, f(x, \cdot) \in L^1(\mathbb{R}^3) \) for \( x \in \Omega \) and \( E, B \in C^1(\overline{\Omega})^3 \) satisfying the
equations (2.1)-(2.9) will be called a stationary solution of the RVMS on $\Omega$. We shall have to relax these conditions in certain situations. We introduce the scalar potential $\Phi$ and the vector potential $A$ by

$$E(x) = -\partial_x \Phi(x), \quad (2.10)$$

$$B(x) = \text{curl} A(x), \quad (2.11)$$

with the Lorentz gauge

$$\text{div} A(x) = 0. \quad (2.12)$$

Then (2.10) implies (2.3), (2.11) implies (2.5), (2.4) is equivalent to

$$-\Delta \Phi(x) = 4\pi \rho(x) \quad (2.13)$$

in view of (2.12) and the well known relation $\text{curl} \text{curl} A = \partial_x \text{div} A - \Delta A$, (2.2) is equivalent to

$$-\Delta A(x) = 4\pi j(x). \quad (2.14)$$

The boundary conditions (2.8) and (2.9) are satisfied if $\Phi$ and $A$ satisfy

$$\Phi(x) = \alpha, \quad x \in \partial \Omega, \quad (2.15)$$

$$\langle \text{curl} A(x), \nu(x) \rangle = 0, \quad x \in \partial \Omega \quad (2.16)$$

for a constant $\alpha \in \mathbb{R}^3$. We observe that if for given $\rho$ and $j$ the potentials $\Phi \in C^4(\overline{\Omega}) \cap C^2(\Omega)$ and $A \in C^4(\overline{\Omega})^3 \cap C^2(\Omega)^3$ satisfy (2.13)-(2.16) then the fields $E, B \in C(\overline{\Omega})^3 \cap C^1(\Omega)^3$ given by (2.10)-(2.11) satisfy (2.2)-(2.9). We find solutions $f$ of (2.1) in three different situations.

**Case 1.** Let $A = 0$. The energy density

$$E(x,v) := \sqrt{1 + v^2 + \Phi(x)}$$

obviously satisfies (2.1). Hence for any $\varphi \in C^1(\mathbb{R})$, $f(x,v) := \varphi(E(x,v))$ is a solution of (2.1). If $f(x,\cdot) \in L^1(\mathbb{R}^3)$ for all $x \in \Omega$, then

$$4\pi \rho(x) = 4\pi \int_{\mathbb{R}^3} f(x,v) \, dv = 4\pi \int_{\mathbb{R}^3} \varphi(\sqrt{1 + v^2 + \Phi(x)}) \, dv = h_\varphi(\Phi(x))$$

with

$$h_\varphi(\xi) := (4\pi)^2 \int_1^\infty \varphi(t + \xi) t \sqrt{t^2 - 1} \, dt. \quad (2.17)$$

Furthermore,

$$j(x) = \int_{\mathbb{R}^3} \dot{v} f(x,v) \, dv = \int_{\mathbb{R}^3} \dot{v} \varphi(\sqrt{1 + v^2 + \Phi(x)}) \, dv = 0,$$

because the integrand is odd in $v$. Hence our choice $A = 0$ is compatible with (2.14) and (2.16). We deduce: To get a stationary solution of the RVMS in Case 1, it is sufficient to solve the problem

$$-\Delta \Phi = h_\varphi(\Phi) \quad \text{in } \Omega,$$

$$\Phi = \alpha \quad \text{on } \partial \Omega \quad (2.18)$$
for $\alpha \geq 0$, with $h_\varphi$ defined by (2.17). A solution will be given by Theorem 3.3 in Section 3.

Case 2. Here we assume $\Omega$ to be cylindrically symmetric, that is, invariant with respect to all rotations about the $x_3$-axis $Z$. We use cylindrical coordinates $(r, \vartheta, z), r(x) := \sqrt{x_1^2 + x_2^2}$, $z := x_3$ for $x \in \overline{\Omega}$. For $x \in \overline{\Omega} \backslash Z$ define the local vector basis

$$e_r(x) := \frac{1}{r(x)} \cdot (x_1, x_2, 0), \quad e_\vartheta(x) := \frac{1}{r(x)} \cdot (-x_2, x_1, 0), \quad e_z(x) := (0, 0, 1).$$

Any vector function $K : \overline{\Omega} \backslash Z \to \mathbb{R}^3$ has a decomposition $K(x) = K_r(x)e_r(x) + K_\vartheta(x)e_\vartheta(x) + K_z(x)e_z(x)$ with

$$K_r(x) := \langle K(x), e_r(x) \rangle, \quad K_\vartheta(x) := \langle K(x), e_\vartheta(x) \rangle, \quad K_z(x) := \langle K(x), e_z(x) \rangle.$$

We define $K$ to be cylindrically symmetric, if $K_r, K_\vartheta, K_z$ are invariant with respect to all rotations about $Z$, that is

$$K_r = K_r(r, z), \quad K_\vartheta = K_\vartheta(r, z), \quad K_z = K_z(r, z)$$

do not depend upon $\vartheta$.

**Lemma 2.1.** If $\Phi$ and $A$ are cylindrically symmetric, then

$$F(x, v) := r(x)(v_\vartheta(x) + A_\vartheta(r, z))$$

is a solution of (2.1) and for $v \in \mathbb{R}^3$, $F(\cdot, v)$ is cylindrically symmetric.

It follows from Lemma 2.1, that for any $\varphi \in C^1(\mathbb{R}^2), \varphi \geq 0$, the function $f := \varphi(\mathcal{E}, F)$ is a solution of (2.1). If $f(x, \cdot) \in L^1(\mathbb{R}^3)$ for all $x \in \overline{\Omega}$, then

$$4\pi \rho(x) = 4\pi \int_{\mathbb{R}^3} f(x, v) \, dv$$

$$= 4\pi \int_{\mathbb{R}^3} \varphi(\sqrt{1 + v^2 + \Phi(x)}, r(x)(v_\vartheta + A_\vartheta(x))) \, dv$$

$$= h_\varphi(r(x), \Phi(x), A_\vartheta(x)),$$

$$4\pi j_\vartheta(x) = 4\pi \int_{\mathbb{R}^3} \vartheta f(x, v) \, dv$$

$$= 4\pi \int_{\mathbb{R}^3} \vartheta \varphi(\sqrt{1 + v^2 + \Phi(x)}, r(x)(v_\vartheta + A_\vartheta(x))) \, dv$$

$$= 8\pi^2 \int_{\mathbb{R}} \frac{v_\vartheta}{\sqrt{1 + v_\vartheta^2 + q^2}} \varphi(\sqrt{1 + v^2 + q^2 + \Phi(x)}, r(x)(v_\vartheta + A_\vartheta(x))) \, dq dv_\vartheta$$

$$= g_\varphi(r(x), \Phi(x), A_\vartheta(x)),$$
with
\[
\begin{align*}
    h_\varphi(r, \xi, \eta) &:= 8\pi^2 \int_0^\infty \int_R^\infty \varphi(\sqrt{1 + v_0^2 + q^2 + \xi, r(v_0 + \eta)}) \, dq \, dv_0, \\
    g_\varphi(r, \xi, \eta) &:= 8\pi^2 \int_0^\infty \int_R^\infty \frac{v_0}{\sqrt{1 + v_0^2 + q^2}} \varphi(\sqrt{1 + v_0^2 + q^2 + \xi, r(v_0 + \eta)}) \, dq \, dv_0
\end{align*}
\]
after introducing cylindrical coordinates \((q = \sqrt{v_r^2 + v_\theta^2}, \gamma, v_\phi)\) with the axis given by \(e_\phi\). The substitution \(t := \pm \sqrt{1 + v_0^2 + q^2}\) yields the unified representation
\[
\begin{align}
    \left(\begin{array}{c}
        h_\varphi \\
        g_\varphi
    \end{array}\right)(r, \xi, \eta) &:= 8\pi^2 \int_{R - \sqrt{t^2 - 1}}^{+\sqrt{t^2 - 1}} \int_R^\infty \varphi(t + \xi, r(s + \eta)) \, ds \, dt. \\
\end{align}
\] (2.19)

Furthermore
\[
\begin{align}
    j_r(x) &= \int_{R^3} \partial_r f(x, v) \, dv \\
    &= \int_{R^3} \partial_r \varphi(\sqrt{1 + v_r^2 + v_\theta^2 + v_\phi^2 + \Phi(x)}, r(x)(v_\phi + A_\phi(x))) \, dv_r \, dv_\theta \, dv_\phi \\
    &= 0
\end{align}
\] (2.20)
because the integrand is odd in \(v_r\), and for a similar reason,
\[
    j_z(x) = 0. \\
\] (2.21)

We now express \(-\Delta A(x) = \text{curl curl} A(x)\) in the system \((e_r(x), e_\phi(x), e_z(x))\). It is well
known that
\[
\begin{align*}
    (\text{curl} A)_r &= \frac{1}{r} \partial_\phi A_z - \partial_z A_\phi, \\
    (\text{curl} A)_\phi &= \partial_z A_r - \partial_r A_z, \\
    (\text{curl} A)_z &= \frac{1}{r} (\partial_r (r A_\phi)) - \frac{1}{r} \partial_\phi A_r.
\end{align*}
\] (2.22)

If \(A\) is also cylindrically symmetric, then \(\partial_\phi A_r = \partial_\phi A_\phi = \partial_\phi A_z = 0\). Hence
\[
\begin{align}
    - (\Delta A)_r &= -\partial_z (\partial_z A_r - \partial_r A_z), \\
    - (\Delta A)_\phi &= \partial_z (\partial_z A_\phi - \partial_r A_\phi) - \partial_r (\frac{1}{r} \partial_r (r A_\phi)) = -\Delta A_\phi + \frac{A_\phi}{r^2}, \\
    - (\Delta A)_z &= \frac{1}{r} (\partial_r (\partial_z A_r - \partial_r A_z)).
\end{align}
\] (2.23)

Let us now choose \(A_r = A_z = 0\) and \(A_\phi\) to be cylindrically symmetric. This is compatible
with (2.12) because
\[
\begin{align*}
    \text{div} A(x) &= \frac{1}{r} \partial_r (r A_r(x)) + \frac{1}{r} \partial_\phi A_\phi(x) + \partial_z A_z(x) = 0.
\end{align*}
\]
As for (2.11) our choice implies
\[
- (\Delta A)_r = 0 = j_r, \quad - (\Delta A)_z = 0 = j_z
\]
with (2.23), (2.20), (2.21). Equation (2.13) and the remaining part of (2.14) now read

\[-\Delta \Phi = h_\varphi (r, \Phi, A_\theta),\]
\[-\Delta A_\theta + \frac{A_\theta}{r^2} = g_\varphi (r, \Phi, A_\theta) \quad \text{in} \quad \Omega.\]

(2.24)

We impose the boundary conditions

\[\Phi = 0, \quad A_\theta = 0 \quad \text{on} \quad \partial \Omega.\]

(2.25)

The fields $E$ and $B$ are then rediscovered from $\Phi$ and $A$ by the general formula

\[(\grad \Phi)_r = \partial_r \Phi, \quad (\grad \Phi)_\theta = \frac{1}{r} \partial_\theta \Phi, \quad (\grad \Phi)_z = \partial_z \Phi \quad \text{for} \quad r \neq 0.\]

(2.26)

and (2.22). If cylindrically symmetric solutions are continuously differentiable near $x \in \partial \Omega \setminus Z$, they satisfy the boundary conditions (2.15), (2.16). In fact, as for (2.16), we note that

\[\grad A_\theta (x) = \partial_r A_\theta (x) \cdot e_r (x) + \partial_\theta A_\theta (x) \cdot e_\theta (x) \]

is a scalar multiple of $\nu (x)$ if $\grad A_\theta (x) \neq 0$, and we get with (2.22) and (2.25) for some $c \in \mathbb{R}$

\[
\langle \curl A (x), \nu (x) \rangle = \frac{1}{cr} A_\theta (x) \partial_z A_\theta (x) = 0.
\]

If $\grad A_\theta (x) = 0$ then $\curl A (x) = 0$ with (2.22) and (2.25).

Note that (2.24) contains a singularity in the term $\frac{A_\theta}{r^2}$ if $\overline{\Omega} \cap Z \neq \emptyset$. In Section 4 we shall first investigate the regular case $\overline{\Omega} \cap Z = \emptyset$ (Theorem 4.2 gives the existence result). The singular case $\overline{\Omega} \cap Z \neq \emptyset$ requires further preparations and will be treated in Section 7 for the case that $\Omega$ is a ball about the origin (see Theorem 7.1).

Case 3. Now let $\Omega$ be translation invariant with respect to $z$. Then we have

Lemma 2.2. If $\Phi$ and $A$ do not depend upon $z$, then

\[P(x, v) := v_z (x) + A_z (x_1, x_2)\]

is a solution of (2.1), and for all $v \in \mathbb{R}^3$, $P$ does not depend upon $z$.

An interesting case arises if $\Omega$ is both cylindrically symmetric and translation invariant, and $\Phi$ and $A$ are cylindrically symmetric and independent of $z$. Then $r$ is the only remaining variable. If $\varphi \in C^1 (\mathbb{R}^3), \varphi \geq 0$, is a given function, then $f := \varphi (E, F, P)$ is a solution of
(2.1). If \( f(x, \cdot) \in L^1(\mathbb{R}^3) \) for all \( x \in \Omega \), then

\[
4\pi \rho(x) = 4\pi \int_{\mathbb{R}^3} f(x, v) \, dv
\]

\[
= 4\pi \int_{\mathbb{R}^3} \varphi(\sqrt{1 + v^2} + \Phi(x), r(x)(v_\theta + A_\theta(x)), v_z + A_z(x)) \, dv
\]

\[
= h_\varphi(r(x), \Phi(x), A_\theta(x), A_z(x)),
\]

\[
4\pi j_\theta(x) = 4\pi \int_{\mathbb{R}^3} \vartheta_\theta f(x, v) \, dv
\]

\[
= 4\pi \int_{\mathbb{R}^3} \vartheta_\theta \varphi(\sqrt{1 + v^2} + \Phi(x), r(x)(v_\theta + A_\theta(x)), v_z + A_z(x)) \, dv
\]

\[
= g_\varphi(r(x), \Phi(x), A_\theta(x), A_z(x)),
\]

\[
4\pi j_z(x) = 4\pi \int_{\mathbb{R}^3} \vartheta_z f(x, v) \, dv
\]

\[
= 4\pi \int_{\mathbb{R}^3} \vartheta_z \varphi(\sqrt{1 + v^2} + \Phi(x), r(x)(v_\theta + A_\theta(x)), v_z + A_z(x)) \, dv
\]

\[
= k_\varphi(r(x), \Phi(x), A_\theta(x), A_z(x)),
\]

where

\[
h_\varphi(r, \xi, \eta, \zeta) := 4\pi \int_{\mathbb{R}^3} \varphi(\sqrt{1 + v^2} + v_\theta^2 + v_z^2 + \xi, r(v_\theta + \eta), v_z + \zeta) \, dv_r \, dv_\theta \, dv_z,
\]

\[
g_\varphi(r, \xi, \eta, \zeta) := 4\pi \int_{\mathbb{R}^3} v_\theta \varphi(\sqrt{1 + v^2} + v_\theta^2 + v_z^2 + \xi, r(v_\theta + \eta), v_z + \zeta) \frac{\sqrt{1 + v_\theta^2 + v_z^2}}{\sqrt{1 + v_\theta^2 + v_z^2}} \, dv_r \, dv_\theta \, dv_z,
\]

\[
k_\varphi(r, \xi, \eta, \zeta) := 4\pi \int_{\mathbb{R}^3} v_z \varphi(\sqrt{1 + v^2} + v_\theta^2 + v_z^2 + \xi, r(v_\theta + \eta), v_z + \zeta) \frac{\sqrt{1 + v_\theta^2 + v_z^2}}{\sqrt{1 + v_\theta^2 + v_z^2}} \, dv_r \, dv_\theta \, dv_z,
\]

and with the substitution \( t := \sqrt{1 + v_\theta^2 + v_z^2} \) one gets the unified representation

\[
\begin{pmatrix}
  h_\varphi \\
g_\varphi \\
k_\varphi
\end{pmatrix}
(r, \xi, \eta, \zeta) = 8\pi \int_{\mathbb{R}^2} \int_{\sqrt{1 + v_\theta^2 + v_z^2}}^\infty \left( \begin{array}{c}
  t \\
v_\theta \\
v_z
\end{array} \right) \varphi(t + \xi, r(v_\theta + \eta), v_z + \zeta) \frac{dt}{\sqrt{t^2 - (1 + v_\theta^2 + v_z^2)}}
\]

\[
\int_{\mathbb{R}^2} d\psi_\theta \, dv_z.
\]

Furthermore, similarly as above

\[
j_r(x) = \int_{\mathbb{R}^3} \vartheta_r f(x, v) \, dv
\]

\[
= \int_{\mathbb{R}^3} \vartheta_r \varphi(\sqrt{1 + v_r^2 + v_\theta^2 + v_z^2} + \Phi(x), r(x)(v_\theta + A_\theta(x)), v_z + A_z(x)) \, dv_r \, dv_\theta \, dv_z
\]

\[
= 0. \quad (2.27)
\]

If we choose \( A_r = 0 \) and \( A_\theta \) and \( A_z \) to depend on \( r \) only, this choice is again compatible.
with (2.12) and it follows from (2.27) and (2.23) that

\((-\Delta A)_r = 0 = j_r\).

Now (2.13) and the two equations remaining in (2.14) read

\[
\begin{align*}
- \Phi'' - \frac{\Phi'}{r} &= h_\Phi(r, \Phi, A_\theta, A_z), \\
- A''_\theta - \frac{A'_\theta}{r^2} + \frac{A_\theta}{r^3} &= g_\Phi(r, \Phi, A_\theta, A_z), \\
- A''_z - \frac{A'_z}{r} &= k_\Phi(r, \Phi, A_\theta, A_z),
\end{align*}
\]

(2.28)

where \((\cdot)'\) denotes differentiation with respect to \(r\). We shall prove the existence of solutions \(\Phi, A_\theta, A_z \in C^2[0, R]\) in Section 6 under the boundary conditions \(\Phi'(0) = A_\theta(0) = A'_z(0) = 0\) in connection with

\[a) \quad \Phi(0) = A'_\theta(0) = A_z(0) = 0 \quad \text{or} \quad b) \quad \Phi(R) = A_\theta(R) = A_z(R) = 0\]

(see Theorem 6.1). Of course, a vanishing derivative at \(r = 0\) guarantees the \(C^2\)-extendability to functions of \(x\) in \(\Omega\); \(\Phi, A_\theta\) and \(A_z\) are constant on \(\partial \Omega\) and hence (2.15) and (2.16) are satisfied (see the above argument).

With the methods presented here, further possibilities could be investigated, e.g. the case \(f = \varphi(\mathcal{E}, P)\). Let us note that our distribution functions have the following interesting property.

**Corollary 2.1.** Any distribution function \(f = f(x, v)\) which only depends upon one or more of the integrals \(\mathcal{E}, F, P\), satisfies the boundary condition of specular reflection at each \(x \in \partial \Omega\), that is,

\[f(x, \tilde{v}(x)) = f(x, v) \quad \text{for} \ \tilde{v}(x) := v - 2\langle v, \nu(x) \rangle \nu(x).\]

In fact, we have \(\tilde{v}(x)^2 = v^2, \tilde{v}_\theta(x) = v_\theta(x)\) (because \(\langle \nu(x), e_\theta(x) \rangle = 0\) in the case of cylindrical symmetry) and \(\tilde{v}_z(x) = v_z(x)\) (in view of \(\langle \nu(x), e_z(x) \rangle = 0\) in the case of translation invariance).

Throughout the paper, universal constants (elements of \(\mathbb{R}\)) will be denoted by \(C\), constants which depend on \(\varphi\) or \(R\) or \(\varphi\) and \(R\), \(\cdots\) will be denoted by \(C_\varphi, C_R, C_{\varphi, R}, \cdots\), and they may vary from line to line.

§3. Distribution Functions Depending Upon \(\mathcal{E}\)

Our solution of problem (2.18) will be based on the following lemma, which is a slightly specialized version of Theorem 9.6 in [1] (p.649). For notations and the assumptions, see also [1] (p.633-634, p.646-647).

**Lemma 3.1 (H. Amann).** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^3\) with boundary \(\partial \Omega \in C^{2+\mu}\) for some \(\mu \in (0, 1)\). Let \(g \in C^{2+\mu}(\partial \Omega), g \geq 0\). Let \(h \in C_\mu(\overline{\Omega} \times [0, \infty)), h(\cdot, 0) \geq 0, \) be such that there is a \(\gamma \geq 0\) with

\[h(x, \xi) - h(x, \xi') \geq -\gamma(\xi - \xi')\]

for all \(x \in \Omega\) and all \(\xi, \xi'\) with \(\xi > \xi' \geq 0\). Assume there is a function \(H \in C(\overline{\Omega})\) and a constant \(\lambda_1 > 0\) such that

\[h(x, \xi) \leq H(x) + \lambda_1 \xi, \quad x \in \overline{\Omega}, \xi \geq 0.\]
Then the boundary value problem

\[-\Delta \Phi = h(x, \Phi) \quad \text{in } \Omega,\]

\[\Phi = g \quad \text{on } \partial \Omega\]

has a minimal nonnegative solution \( \Phi \in C^2(\Omega) \) provided \( \lambda_1 < \lambda_0 \), where \( \lambda_0 \) denotes the smallest (positive) eigenvalue of the linear eigenvalue problem

\[-\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.\]

We are going to define a class of functions \( \varphi \) in Case 1 so that the associated function \( h_\varphi \) defined in (2.17) satisfies the assumptions of Lemma 3.1.

**Lemma 3.2.** Let \( \varphi \in C^1[1, \infty) \) be nonnegative and satisfy the following two conditions:

i) \( \forall \xi \geq 0 : (t \mapsto \varphi(t + \xi)t^{\sqrt{t^2} - 1} \in L^1(1, \infty), \)

ii) \( \exists m \in L^1(1, \infty) \forall \xi \geq 0 \forall t \geq 1 : |\varphi'(t + \xi)t^{\sqrt{t^2} - 1}| \leq m(t). \)

Then \( h_\varphi \), given by

\[ h_\varphi(\xi) := (4\pi)^2 \int_1^\infty \varphi(t + \xi)t^{\sqrt{t^2} - 1} dt, \quad \xi \geq 0 \]

is nonnegative, monotonically decreasing to zero for \( \xi \to \infty \) and continuously differentiable on \([0, \infty)\) with bounded derivative

\[ h'_\varphi(\xi) := (4\pi)^2 \int_1^\infty \varphi'(t + \xi)t^{\sqrt{t^2} - 1} dt, \quad \xi \geq 0. \]

The proof of Lemma 3.2 is straightforward. Lemmas 3.1 and 3.2 yield

**Theorem 3.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with boundary \( \partial \Omega \in C^{2+\mu} \) for some \( \mu \in (0, 1) \). Let \( \varphi \in C^1[1, \infty) \) satisfy the assumptions of Lemma 3.2 and let \( \alpha \geq 0 \). Then the problem

\[-\Delta \Phi = h_\varphi(\Phi) \quad \text{in } \Omega,\]

\[\Phi = \alpha \quad \text{on } \partial \Omega\]

has a nonnegative solution \( \Phi \in C^2(\Omega) \). Consequently, every such \( \varphi \) induces a stationary solution \((f, E, B)\) of the RVMS on \( \Omega \) such that \( f = \varphi(E), E = -\partial_x \Phi \) and \( B = 0 \).

**Proof.** We may apply Lemma 3.2 to see that \( h_\varphi \) satisfies the conditions of Lemma 3.1. In fact, we may define \( \gamma := \sup_{\xi \geq 0} |h'_\varphi(\xi)|, H(x) := h_\varphi(0) \) and \( \lambda_1 = 0, \) and the assertion follows from Lemma 3.1.

**§4. Distribution Functions Depending Upon \( \varepsilon \) and \( P \) (Regular Case)**

In this section we are going to solve the system (2.24) with the boundary conditions (2.25). By applying methods and theorems of nonlinear functional analysis in ordered Banach spaces it is possible to generalize Lemma 3.1 to an existence theorem for semilinear elliptic systems (see [1, p.654]). However, the main assumption is that the right hand side of the system has to be increasing in the "off-diagonal" variables, and there do not seem to exist examples \( \varphi \) for which the resulting right hand side \( (h_\varphi, g_\varphi) \) would satisfy this condition. The following
theorem essentially goes back to P.J. McKenna and W. Walter\textsuperscript{[15,p.209]}. The version given here asserts a slightly stronger regularity of the solution which we need later. Because not all details of the proof can be found in [15] and because it will be necessary to conclude the existence of cylindrically symmetric solutions in Theorem 4.2 we are going to provide the main arguments here. In the following, inequalities between vectors in $\mathbb{R}^n$ are to be understood componentwise.

**Theorem 4.1** (P.J. McKenna - W. Walter). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain ($n \in \mathbb{N}$) with boundary $\partial \Omega \in C^{2+\mu}$ for some $\mu \in (0,1)$. Let $F: \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following condition: For all $\eta > 0$ there exists $C_\eta > 0$ such that for all $x, x_i \in \overline{\Omega}$, $y, y_i \in \mathbb{R}^n$ with $|y|, |y_i| \leq \eta$, $i = 1, 2$ one has

$$|F(x_1, y) - F(x_2, y)| \leq C_\eta|x_1 - x_2|^\mu,$$

$$|F(x, y_1) - F(x, y_2)| \leq C_\eta|y_1 - y_2|.$$

Assume further that there exists a pair of vector functions $v, w \in C^1(\Omega)^n \cap C^2(\Omega)^n$ with $v \leq w$ in $\overline{\Omega}$ and $v(0) = 0$ on $\partial \Omega$, such that for $i = 1, \cdots, n$:

$$\forall x \in \Omega, \forall z \in \mathbb{R}^n, v(x) \leq z \leq w(x), z_i = v_i(x) : -\Delta v_i(x) \leq F_i(x, z),$$

$$\forall x \in \Omega, \forall z \in \mathbb{R}^n, v(x) \leq z \leq w(x), z_i = w_i(x) : -\Delta w_i(x) \geq F_i(x, z).$$

Then there exists a solution $u \in C^{2+\mu}(\overline{\Omega})^n$ of the problem

$$-\Delta u = F(x, u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

and $v \leq u \leq w$ pointwise on $\overline{\Omega}$.

**Proof.** 1. We introduce a cut-off $P: \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$P_i(x, z) \mapsto \begin{cases} w_i(x) & \text{if } z_i \geq w_i(x), \\ v_i(x) & \text{if } z_i \leq v_i(x), \\ z_i & \text{else.} \end{cases}$$

and let

$$G(x, z) := F(x, P(x, z)) + \arctan(P(x, z) - z)$$

(we define $\arctan z := (\arctan z_i)_{i=1..n}$ for $z \in \mathbb{R}^n$). Then $G$ is bounded, has the same regularity properties as $F$, and $G(x, \cdot) = F(x, \cdot)$ on $[w(x), v(x)]$.

2. We prove the existence of a solution $u \in C^{2+\mu}(\overline{\Omega})^n$ of $-\Delta u = G(x, u)$ in $\Omega$, $u = 0$ on $\partial \Omega$. Define the Nemytskii-Operator

$$\tilde{G}: C(\Omega)^n \to C(\Omega)^n \quad \text{by } \psi \mapsto G(\cdot, \psi(\cdot)).$$

In particular,

$$\tilde{G}: C^1(\overline{\Omega})^n \to C^\mu(\overline{\Omega})^n$$

and $\tilde{G}$ maps bounded sets into bounded sets\textsuperscript{[1,p.647]}. The inverse $(-\Delta)^{-1}: C^\mu(\overline{\Omega})^n \to C^{2+\mu}(\overline{\Omega})^n$ is defined with respect to zero boundary values\textsuperscript{[1,p.635] and has a unique extension to a compact operator $C(\Omega)^n \to C^\sigma(\Omega)^n$ for all $\sigma \in [0, 2)$, again denoted by $(-\Delta)^{-1}$ (see \textsuperscript{[1, p.635]}). (***)

Hence $K := (-\Delta)^{-1} \circ \tilde{G}: C(\Omega)^n \to C(\Omega)^n$ is compact. $\tilde{G}$ maps $C(\Omega)^n$ into some open ball $B$ about 0. The mapping $\lambda \mapsto \text{id} - \lambda K$ on $[0, 1]$ is a homotopy on $[0, 1]$, and
0 \notin (\text{id} - \lambda K)(\partial B). \text{ Hence the Leray-Schauder degree } \text{ } D(\text{id} - \lambda K, B, 0) = 1. \text{ There exists } u \in C(\overline{\Omega})^n \text{ with } u = \text{Ku. By (}), \tilde{G}u \in C(\overline{\Omega})^n. \text{ By (***)}, u = \text{Ku} \in C^1(\overline{\Omega})^n. \text{ By (**)}, \tilde{G}u \in C^2(\overline{\Omega})^n, \text{ so that } u = \text{Ku} \in C^2(\overline{\Omega})^n.

3. \text{ We show } v \leq u \leq w. \text{ In fact, assume there exists } i \in \{1, \cdots, n\} (\text{ we let } i = 1) \text{ such that } \min(w_i - u_i) < 0. \text{ There exists } x_0 \in \Omega \text{ such that the minimum is given by } w_1(x_0) - u_1(x_0) \text{ (for } x_0 \in \partial\Omega \text{ implies } w_1(x_0) \geq 0 \text{ and } u_1(x_0) = 0). \text{ Then, for } \delta := (P_1(x_0, u(x_0)), \cdots, P_n(x_0, u(x_0))) \text{ we have }
\begin{align*}
0 &\leq \Delta(w_1 - u_1)(x_0) = \Delta w_1(x_0) + G_1(x_0, u(x_0)) \\
&\leq G_1(x_0, u(x_0)) - F_1(x_0, (w_1(x_0), \delta)) \\
&= (F_1(x_0, P(x_0, u(x_0))) - F_1(x_0, (w_1(x_0), \delta))) + \arctan(P_1(x_0, u(x_0)) - u_1(x_0)) \\
&< 0,
\end{align*}

which is a contradiction. The inequality } v \leq u \text{ is proven similarly. It follows that } u \text{ is the desired solution.}

We define } C'_{cy} := \{ f \in C(\overline{\Omega}) : f \circ R = f \text{ for all rotations } R \text{ about } Z}\).

**Corollary 4.1.** \text{ In addition to the hypotheses of Theorem 4.1 we assume for } n = 3: \Omega \text{ is cylindrically symmetric and } F(\cdot, y) \in C'_{cy}(\overline{\Omega}) \text{ for all } y \in \mathbb{R}^3 \text{ and } v, w \in C'_{cy}(\overline{\Omega}). \text{ Then there exists a solution } u \in C'_{cy}(\overline{\Omega})^3 \cap C^{2+\mu}(\overline{\Omega})^3 \text{ such that } v \leq u \leq w.

**Proof.** \text{ We observe that the class } C'_{cy}(\overline{\Omega}) \text{ is a closed subspace of } C(\overline{\Omega}) \text{ and that the arguments of the proof of Theorem 4.1 can be carried through in } C'_{cy}(\overline{\Omega}).

We now make our choice of suitable functions } \varphi \text{ which allow an application of the foregoing results.}

**Lemma 4.1.** \text{ Let } \varphi \in C^1([1, \infty) \times \mathbb{R}) \text{ be nonnegative and satisfy the following condition: } \exists m \in L^1([1, \infty) \times \mathbb{R}) \forall \epsilon \geq 1 \forall F \in \mathbb{R}:
\begin{align*}
\varphi(\epsilon, F)\epsilon \sqrt{\epsilon^2 - 1} &\leq m(\epsilon), \\
|\partial_{\epsilon}\varphi(\epsilon, F)|\epsilon \sqrt{\epsilon^2 - 1} &\leq m(\epsilon), \\
|\partial_F\varphi(\epsilon, F)|\epsilon (\epsilon^2 - 1) &\leq m(\epsilon).
\end{align*}

Then the functions } h_\varphi, g_\varphi \text{ given by }
\begin{align*}
(h_\varphi, g_\varphi)(x, \xi, \eta) = 8\pi^2 \int_1^{\infty} \int_{\sqrt{t^2 - 1}}^{\infty} \left( \int_{-\sqrt{t^2 - 1}}^{t} \varphi(t + \xi, \gamma)\text{d}s \right) d\gamma \text{d}t, \quad x \in \mathbb{R}^3, \xi \geq 0, \eta \in \mathbb{R}
\end{align*}

are continuous on } \mathbb{R}^3 \times [0, \infty) \times \mathbb{R} \text{ together with their derivatives with respect to } \xi \text{ and } \eta, \partial_\xi h_\varphi \text{ and } \partial_\xi g_\varphi \text{ exist as continuous functions on } \mathbb{R}^3 \setminus Z \times [0, \infty) \times \mathbb{R}. \text{ We have the following estimates:}
\begin{align*}
0 &\leq \frac{1}{2} h_\varphi, |g_\varphi|, \frac{1}{2} |\partial_\xi h_\varphi|, |\partial_\xi g_\varphi| \leq 8\pi^2 \| m \|_1, \\
\frac{1}{2} |\partial_\eta h_\varphi(x, \xi, \eta)|, |\partial_\eta g_\varphi(x, \xi, \eta)| &\leq 8\pi^2 \| m \|_1 \cdot r(x), \\
\text{ and for } r(x) > 0: \frac{1}{2} |\partial_x h_\varphi(x, \xi, \eta)|, |\partial_x g_\varphi(x, \xi, \eta)| &\leq 8\pi^2 \| m \|_1 \cdot (1 + |\eta|).
\end{align*}

Furthermore, } g_\varphi(0, \xi, \eta) = 0.
Proof. We note
\[ h_\varphi(x, \xi, \eta) \leq 8\pi^2 \int_1^{\sqrt{t^2-1}} \int t \frac{m(t + \xi)}{(t + \xi) \sqrt{(t + \xi)^2 - 1}} \, ds \, dt \]
\[ = 8\pi^2 \int_1^{\sqrt{t^2-1}} 2t \sqrt{t^2 - 1} \frac{m(t + \xi)}{(t + \xi) \sqrt{(t + \xi)^2 - 1}} \, dt \]
\[ \leq 16\pi^2 \int_1^\infty m(t + \xi) \, dt \leq 16\pi^2 \| m \|_1, \]
\[ |\partial_x h_\varphi(x, \xi, \eta)| = 8\pi^2 \int_1^{\sqrt{t^2-1}} \int t(s + \eta) \partial_x \varphi(t + \xi, r(x)(s + \eta)) \, ds \, dt \cdot e_r(x) \]
\[ \leq 8\pi^2 \| \partial_x \varphi \|_1 + 8\pi^2 \int_1^\infty t(t^2 - 1) \frac{m(t + \xi)}{(t + \xi) \sqrt{(t + \xi)^2 - 1}} \, ds \, dt \]
\[ \leq 8\pi^2 (|\eta| + 1) \| m \|_1 \text{ for } r(x) > 0. \]
The other estimates are similar.

We can now treat the regular case of the system (2.24), (2.25), in which \( \overline{\Omega} \) does not contain points of the z-axis \( Z \).

Theorem 4.2. Let \( \Omega \subset \mathbb{R}^3 \) be a cylindrically symmetric bounded domain with \( \partial \Omega \subset C^{2+\mu} \) for some \( \mu \in (0,1) \) and assume \( \overline{\Omega} \cap Z = \emptyset \). Let \( \varphi \in C^1([1,\infty) \times \mathbb{R}) \) satisfy the assumptions of Lemma 4.1. Then the problem
\[ -\Delta \Phi = h_\varphi(r, \varphi, A_\varphi), \]
\[ -\Delta A_\varphi = g_\varphi(r, \varphi, A_\varphi) - \frac{A_\varphi}{r^2} \quad \text{in } \Omega, \]
\[ \Phi = 0, \quad A_\varphi = 0 \quad \text{on } \partial \Omega, \]
is a cylindrically symmetric solution \( \Phi, A_\varphi \in C^{2+\mu}(\overline{\Omega}) \). Consequently, every such \( \varphi \) induces a stationary solution \( (f, E, B) \) of the RVMS on \( \Omega \) such that \( f = \varphi(E, F) \), \( E = -\partial_x \Phi \) and \( B = \text{curl} A \) with \( A_r = A_z = 0 \).

Proof. It follows from Lemma 4.1 that the right hand side of the system (4.1) satisfies the regularity assumptions of Theorem 4.1. Our next concern is the construction of the sub- and supersolution \( v \) and \( w \) assumed in Theorem 4.1.

Let \( R := \max_{x \in \Omega} r(x) \). We know from Lemma 5.2
\[ 0 \leq h_\varphi \text{ and } \frac{1}{2} \| h_\varphi \|_1, \| g_\varphi \| \leq 8\pi^2 \| m \|_1 < \infty. \]
We solve the boundary value problems
\[ -\frac{1}{r} (rv'_1)' = 0, \quad v'_1(0) = v_1(R) = 0, \]
\[ -\frac{1}{r} (rw'_1)' = \| h_\varphi \|, \quad w'_1(0) = w_1(R) = 0, \]
\[-\frac{1}{r}(rv_2)' + \frac{v_2}{r^2} = -\|g_\varphi\|, \quad v_2(0) = v_2(R) = 0,\]
\[-\frac{1}{r}(rw_2)' + \frac{w_2}{r^2} = \|g_\varphi\|, \quad w_2(0) = w_2(R) = 0\]

and get

\[v_1 = 0, \quad w_1(r) = \frac{1}{4} \|h_\varphi\| (R^2 - r^2), \quad w_2(r) = \frac{1}{3} \|h_\varphi\| r(R - r), \quad v_2 = -w_2\]

(see Lemmas 5.1 and 5.2). We may define \(v := (v_1, v_2), w := (w_1, w_2)\). In fact, we have

\[v \leq 0 \leq w\]

and for all \(z \in \mathbb{R}^2\) with \(z_1 \geq 0\)

\[-\Delta v_1(x) = 0 \leq h_\varphi(x, z),\]
\[-\Delta v_2(x) = -\|g_\varphi\| - \frac{v_2(x)}{r^2} \leq g_\varphi(x, z) - \frac{z_2}{r^2}, \quad \text{if} \quad z_2 = v_2(x)\]
\[-\Delta w_1(x) = \|h_\varphi\| \geq h_\varphi(x, z),\]
\[-\Delta w_2(x) = -\|g_\varphi\| - \frac{w_2(x)}{r^2} \geq g_\varphi(x, z) - \frac{z_2}{r^2}, \quad \text{if} \quad z_2 = w_2(x).\]

The existence of a solution in \(C_{\text{curl}}(\Omega)^2 \cap C^{2+\mu}(\Omega)^2\) now follows from Corollary 4.1.

§5. Explicit Solutions of Particular Singular Second Order Equations

In this section we collect some results on certain ordinary differential equations of second order with singular coefficients at \(r = 0\). The statements made will be needed in the following two sections. Our general assumption is \(f \in C([0, R])\) for some \(R > 0\). For \(a, b \in \mathbb{R}\), we let \([a < s < b]\) be the characteristic function (in \(s\)) of the interval \([a, b]\).

**Lemma 5.1.** Consider the equation

\[u'' + \frac{u'}{r} = f(r), \quad 0 < r \leq R.\]

a) The solution \(u_0 \in C^2[0, R]\) with \(u_0(0) = u_0'(0) = 0\) is given by

\[u_0(r) = \int_0^r \int_0^s f(s) \, ds \, ds = \int_0^r \int_0^s \frac{r}{s} f(s) \, ds.\]

We have \(u_0''(0) = \frac{1}{2} f(0)\).

b) The general solution \(u \in C^2(0, R]\) is

\[u(r) = a_1 + a_2 \log r + u_0(r), \quad a_1, a_2 \in \mathbb{R}.\]

The following conditions are equivalent:

i) \(u\) or \(u'\) is bounded at \(r = 0\) or has a finite limit for \(r \to 0\),

ii) \(\lim_{r \to 0^+} u'(r) = 0\),

iii) \(\lim_{r \to 0^+} r \cdot u'(r) = 0\),

iv) \(a_2 = 0\).

In this case \(u \in C^2[0, R]\), \(u''(0) = \frac{1}{2} f(0)\); and \(u(R) = 0\) iff \(u\) is

\[u_R(r) = -u_0(R) + u_0(r)\]

\[= \int_0^R \left\{ \frac{s}{s \log \frac{R}{s}} \right\} f(s) \, ds.\]
\textbf{Lemma 5.2.} Consider the equation
\begin{equation}
v'' + \frac{v'}{r} - \frac{v}{r^2} = f(r), \quad 0 < r \leq R. \tag{5.1}
\end{equation}

a) The solution \( v_0 \in C^2[0, R] \) with \( v_0(0) = v'_{0}(0) = 0 \) is given by
\begin{equation}
v_0(r) = \frac{1}{r} \int_0^r \int_0^s f(\sigma) d\sigma ds = \frac{1}{2} \int_0^R \left\{ \left[ s \leq r \right] \left( 1 - \frac{r^2}{2s^2} \right) r f(s) ds \right\}.
\end{equation}
We have \( v'_0(0) = \frac{2}{3} f(0) \).

b) The general solution \( v \in C^2(0, R) \) is
\begin{equation}
v(r) = b_1 r + b_2 \frac{1}{r} + v_0(r), \quad b_1, b_2 \in \mathbb{R}.
\end{equation}

The following conditions are equivalent:
\begin{enumerate}
\item \( v \) or \( v' \) is bounded at \( r = 0 \) or has a finite limit for \( r \to 0 \),
\item \( \lim_{r \to 0} v(r) = 0 \),
\item \( \lim_{r \to 0} r \cdot v(r) = 0 \),
\item \( b_2 = 0 \).
\end{enumerate}
In this case \( v \in C^2[0, R] \), \( v''(0) = -\frac{1}{3} f(0) \); and \( v(R) = 0 \) iff \( v \) is
\begin{equation}
v_R(r) = -v_0(R) r \frac{1}{R} + v_0(r)
\end{equation}

\begin{equation}
= -\frac{1}{2} \int_0^R \left\{ \left[ s \leq r \left( 1 - \frac{r}{2s^2} \right) s^2 + \left[ s \geq r \right] \left( 1 - \frac{s^2}{2r^2} \right) r \right] f(s) ds \right\}.
\end{equation}

(For \( f(r) = \lambda \) one has \( u_R(r) = -\frac{\lambda}{4}(R^2 - r^2) \).)

\textbf{Lemma 5.3.} For \( 0 < \delta \leq R \) consider the equation
\begin{equation}
w'' + \frac{w'}{r} - \frac{w}{\delta^2} = f(r), \quad 0 < r \leq R. \tag{5.2}
\end{equation}

a) The solution \( w_0 \in C^2[0, R] \) with \( w_0(0) = w'_0(0) = 0 \) is given by
\begin{equation}
w_0(r) = z_\delta(r) \int_0^r \left( I_\delta(r) - I_\delta(s) \right) z_\delta(s) s f(s) ds,
\end{equation}
where
\begin{equation}
z_\delta(r) := \sum_{k=0}^{\infty} \frac{(r/\delta)^{2k}}{[(2k)!!]^2}, \quad I_\delta(r) := \int_\delta^r \frac{ds}{sz_\delta^2(s)}.
\end{equation}

Here \( z_\delta \in C^2[0, R] \) is a solution of the homogeneous equation with
\begin{equation}
z_\delta(0) = 1, \quad z'_\delta(0) = 0, \quad z''_\delta(0) = \frac{1}{2\delta^2},
\end{equation}
\begin{equation}
z_\delta(\delta) = S, \quad z'_\delta(\delta) = \frac{S_1}{\delta},
\end{equation}
with \( S := \sum_{k=0}^{\infty} [(2k)!!]^{-2}, \quad S_1 := \sum_{k=1}^{\infty} 2k[(2k)!!]^{-2} \).

We have \( w'_0(0) = \frac{3}{3} f(0) \).
b) The general solution \( w \in C^2(0, R) \) is
\[
w(r) = c_1 z_\delta(r) + c_2 I_\delta(r) z_\delta(r) + w_0(r), \quad c_1, c_2 \in \mathbb{R}.
\]
The following conditions are equivalent:

i) \( w \) or \( w' \) is bounded at \( r = 0 \) or has a finite limit for \( r \to 0 \),

ii) \( \lim_{r \to 0} w'(r) = 0 \),

iii) \( c_2 = 0 \).

In this case \( w \in C^2[0, R] \), \( w''(0) = \frac{\partial^2 f}{\partial r^2} + f(0) \).

Now we define
\[
r_\delta(r) := \begin{cases} 
\delta & \text{for } 0 \leq r \leq \delta \\
r & \text{for } r \geq \delta. 
\end{cases}
\]  
(5.3)

Lemma 5.4. For \( 0 < \delta \leq R \) the solution \( v_\delta \in C^2[0, R] \) of
\[
\frac{v''}{r} + \frac{v'}{r} - \frac{v}{r^2} = f(r), \quad 0 < r \leq R
\]  
(5.4)

with \( v_\delta(R) = 0 \) is given by
\[
v_\delta(r) = c_\delta z_\delta(r) + w_0(r)
\]
\[
= z_\delta(r) \left(c_\delta + \int_0^r (I_\delta(r) - I_\delta(s)) z_\delta(s) s f(s) ds\right), \quad 0 \leq r \leq \delta,
\]
\[
v_\delta(r) = b_\delta \left(\frac{1}{r} - \frac{r}{R^2}\right) + v_R(r), \quad \delta \leq r \leq R,
\]
where \( w_0 \) and \( v_R \) are defined in Lemma 5.3 and 5.2 respectively, and
\[
c_\delta := -\frac{1}{N} \int_0^\delta \left\{ \int_0^r \left[ s \leq \delta \right] \frac{R^2 - \delta^2}{S} - z_\delta(s)s + [s \geq \delta] (R^2 - s^2) \delta \right\} f(s) ds
\]
\[
- \int_0^\delta (I_\delta(\delta) - I_\delta(s)) z_\delta(s) s f(s) ds,
\]  
(5.5)

\[
b_\delta := -\frac{1}{2} \int_0^R \left\{ \int_0^r \left[ s \leq \delta \right] \frac{2R^2}{N} \delta z_\delta(s) - s \right\}
\]
\[
+ [s \geq \delta] \frac{S - S_1}{N} (R^2 - s^2) \delta^2 \right\} f(s) ds,
\]  
(5.6)

with \( N := (S + S_1)R^2 + \delta^2(S - S_1) \).

Proof. Lemma 5.3 implies that \( c z_\delta + w_0, c \in \mathbb{R} \), is the general bounded solution of (5.2), and Lemma 5.2 says that \( b(\frac{1}{r} - \frac{r}{R^2}) + v_R, b \in \mathbb{R} \), is the general solution of (5.1) which vanishes at \( r = R \). We can determine the constants in such a way that the solutions and their first derivatives have the same value at \( r = \delta \) and thus obtain the formulas (5.5), (5.6). Because \( r_\delta \) is continuous this implies the continuity of the second derivative at \( r = \delta \) and \( v_\delta \in C^2[0, R] \) follows.

Corollary 5.1. There exists a constant \( C^*_R > 0 \) (only depending upon \( R \)) such that for
all \( f \in C[0,R] \) with \( f \leq 0 \) one has

\[
0 \leq v_R(r), v_\delta(r) \leq C_R^\delta \| f \| \delta, \quad 0 \leq r \leq \delta,
\]

\[
0 \leq v_R(r) \leq v_\delta(r) \leq v_R(r) + C_R^\delta \| f \| \delta, \quad \delta \leq r \leq R,
\]

for small \( \delta > 0 \).

**Proof.** The representation of \( v_R \) in Lemma 5.2 b) implies \( v_R \geq 0 \) on \([0,R]\) and \( v_R \leq C_R^\delta \| f \| \delta \) on \([0,\delta]\). The relations

\[
\int_0^r (I_\delta(r) - I_\delta(s)) z_\delta(s) s \, ds = O(\delta^2) \quad (\delta \to 0)
\]

uniformly for \( r \leq \delta \),

\[
c_\delta = O(\delta \| f \|), \quad b_\delta = O(\delta^2 \| f \|) \quad (\delta \to 0)
\]

are obvious. The integral kernel which represents \(-b_\delta\) is nonnegative because for \( s \leq \delta\)

\[
2R^2 \delta z_\delta(s) - N s \geq R^2 \delta \left( 2 - \left[ S + S_1 + \frac{S - S_1}{R^2} \delta^2 \right] \right)
\]

> 0

for all \( \delta > 0 \) such that \( S + S_1 + \frac{S - S_1}{R^2} \delta^2 < 2 \) (note that \( S + S_1 < 2 \) and \( S - S_1 > 0 \)). Hence \( b_\delta \geq 0 \) and because \( c_\delta \geq 0 \) the representation of \( v_\delta \) in Lemma 5.4 gives the result.

### §6. Distribution Functions Depending Upon \( \varepsilon, F \) and \( P \)

This section is devoted to the study of the system (2.28). The following lemma gives sufficient conditions on \( \varphi \) such that existence can be proven later by Schauder’s fixed point theorem.

**Lemma 6.1.** Let \( \varphi \in C^1([1,\infty) \times \mathbb{R} \times [0,\infty)) \) be nonnegative and satisfy the following condition: \( \exists m \in L^1(1,\infty) \forall \varepsilon \geq 1 \forall F \in \mathbb{R} \forall P \geq 0 : \)

\[
\varphi(\varepsilon, F, P) \varepsilon \sqrt{\varepsilon^2 - 1} \leq m(\varepsilon),
\]

\[
|\partial_\varepsilon \varphi(\varepsilon, F, P)| \varepsilon \sqrt{\varepsilon^2 - 1}, |\partial_F \varphi(\varepsilon, F, P)| \varepsilon \sqrt{\varepsilon^2 - 1} \leq m(\varepsilon),
\]

\[
|\partial_P \varphi(\varepsilon, F, P)| (\varepsilon \sqrt{\varepsilon^2 - 1}) \leq m(\varepsilon).
\]

Then the functions \( h_\varphi, g_\varphi, k_\varphi \) given by

\[
\begin{pmatrix} h_\varphi \\ g_\varphi \\ k_\varphi \end{pmatrix} (x, \xi, \eta, \zeta) := 8\pi \int_{\mathbb{R}^2} \int_0^\infty \begin{pmatrix} t \\ v_\eta \\ v_\zeta \end{pmatrix} \frac{\varphi(t + \xi, r(\eta + r), v_\eta + \zeta)}{\sqrt{t^2 - (1 + v_\eta^2 + v_\zeta^2)}} \, dtdv_\eta dv_\zeta
\]

are continuous on \( \mathbb{R}^3 \times [0,\infty) \times \mathbb{R}^2 \) together with their derivatives with respect to \( \xi, \eta \) and \( \zeta; \partial_\xi h_\varphi, \partial_\eta g_\varphi \) and \( \partial_\zeta k_\varphi \) exist as continuous functions on \( \mathbb{R}^3 \setminus \mathbb{Z} \times [0,\infty) \times \mathbb{R}^2 \). We have the following estimates:

\[
0 \leq \frac{1}{2} |h_\varphi|, |g_\varphi|, |k_\varphi|, \frac{1}{2} |\partial_\xi h_\varphi|, |\partial_\xi g_\varphi|, |\partial_\xi k_\varphi|,
\]

\[
\frac{1}{2} |\partial_\eta h_\varphi|, |\partial_\eta g_\varphi|, |\partial_\eta k_\varphi| \leq 8\pi^2 \| m \|_1,
\]

\[
\frac{1}{2} |\partial_\zeta h_\varphi|, |\partial_\zeta g_\varphi|, |\partial_\zeta k_\varphi| \leq 8\pi^2 \| m \|_1 r(x),
\]
and for \( r(x) > 0 \):
\[
\frac{1}{2} |\partial_x h_\varphi(x, \xi, \eta, \zeta)|, |\partial_x g_\varphi(x, \xi, \eta, \zeta)|, |\partial_x k_\varphi(x, \xi, \eta, \zeta)| \leq 8\pi^2 \| m \|_1 (1 + |\eta|).
\]
Furthermore, \( g_\varphi(0, \xi, \eta, \zeta) = 0 \).

**Proof.** For \( a > 0 \) and \( k = 0, 1, 2, 3 \) we have
\[
I_k(a) := \int_0^a \frac{s^k}{\sqrt{a^2 - s^2}} ds = a^k I_k(1),
\]
where \( I_0(1) = \frac{\pi}{2}, I_1(1) = 1, I_2(1) = \frac{\pi}{4}, I_3(1) = \frac{2}{3} \). Hence
\[
0 \leq h_\varphi(x, \xi, \eta, \zeta) \leq 8\pi \int_{\mathbb{R}^2} \int_0^\infty \frac{m(t + \xi)}{(t + \xi)\sqrt{(t + \xi)^2 - 1} \sqrt{t^2 - (1 + v_\varphi^2 + v_\zeta^2)}} \frac{t}{\sqrt{t^2 - (1 + v_\varphi^2 + v_\zeta^2)}} dt dv_\varphi dv_\zeta
\]
\[
= 16\pi^2 \int_0^\infty \int_0^\infty \frac{m(t + \xi)}{(t + \xi)\sqrt{(t + \xi)^2 - 1} \sqrt{t^2 - (1 + \rho^2)}} \frac{t}{\sqrt{t^2 - (1 + \rho^2)}} d\rho dt
\]
\[
= 16\pi^2 \int_1^\infty \frac{m(t + \xi)}{(t + \xi)\sqrt{(t + \xi)^2 - 1} \sqrt{t^2 - 1}} \frac{\rho}{\sqrt{t^2 - 1}} d\rho dt
\]
\[
\leq 16\pi^2 \| m \|_1 .
\]
Similarly, with the substitution \( \sigma := v_\varphi, \rho := \sqrt{v_\varphi^2 + v_\zeta^2}, \)
\[
|g_\varphi(x, \xi, \eta, \zeta)| \leq 32\pi \int_0^\infty \int_0^\infty \int_0^\infty \frac{m(t + \xi)}{(t + \xi)\sqrt{(t + \xi)^2 - 1} \sqrt{t^2 - (1 + v_\varphi^2 + v_\zeta^2)}} \frac{v_\varphi}{\sqrt{t^2 - (1 + v_\varphi^2 + v_\zeta^2)}} dv_\varphi dv_\zeta dt
\]
\[
= 32\pi \int_0^\rho \int_0^\rho \int_0^\rho \frac{m(t + \xi)}{(t + \xi)\sqrt{(t + \xi)^2 - 1} \sqrt{t^2 - (1 + \rho^2)}} \frac{\sigma}{\sqrt{t^2 - (1 + \rho^2)}} d\sigma d\rho d\rho
\]
\[
= 32\pi \int_0^\rho \int_0^\rho \frac{m(t + \xi)}{(t + \xi)\sqrt{(t + \xi)^2 - 1} \sqrt{t^2 - (1 + \rho^2)}} \frac{\rho}{\sqrt{t^2 - (1 + \rho^2)}} d\rho dt
\]
\[
= 32\pi \int_1^\infty \frac{m(t + \xi)}{(t + \xi)\sqrt{(t + \xi)^2 - 1} \sqrt{t^2 - 1}} \frac{\rho^2}{\sqrt{t^2 - 1}} d\rho dt
\]
\[
\leq 8\pi^2 \| m \|_1 .
\]
The remaining estimates follow in a similar way.

We remark that a radial function defined on an interval \([0, a]\) is \( C^2 \) on a neighborhood of
in $\mathbb{R}^n$ if it belongs to $C^2[0,a]$ and its radial derivative vanishes at $r = 0$.

**Theorem 6.1.** Let $\Omega \subset \mathbb{R}^3$ be a cylindrical domain of the form $\Omega = \{x \in \mathbb{R}^3 : r(x) < R\}$ for some $R > 0$. Let $\varphi \in C^1([1, \infty) \times \mathbb{R} \times [0, \infty))$ satisfy the assumptions of Lemma 6.1. Then the system

$$
\begin{align*}
-\Phi'' - \frac{\Phi'}{r} &= h_\varphi(r, \Phi, A_\theta, A_z) \\
-A''_\theta - \frac{A'_\theta}{r^2} &= g_\varphi(r, \Phi, A_\theta, A_z) \\
-A''_z - \frac{A'_z}{r} &= k_\varphi(r, \Phi, A_\theta, A_z),
\end{align*}
$$

(6.1)

in connection with $\Phi'(0) = A_\theta(0) = A_z'(0) = 0$ and

a) $\Phi(0) = A'_\theta(0) = A_z(0) = 0$ or b) $\Phi(R) = A_\theta(R) = A_z(R) = 0$

has a solution $(\Phi, A_\theta, A_z) \in C^2[0,R]^3$. Consequently, every such $\varphi$ induces a stationary solution $(f, E, B)$ such that $f = \varphi(\xi, F, P)$; $E, B$ only depend upon $r$, and $f \in C((\bar{\Omega} \setminus \mathbb{Z}) \times \mathbb{R}^3) \cap C^1((\bar{\Omega} \setminus \mathbb{Z}) \times \mathbb{R}^3)$, $E, B \in C^1(\bar{\Omega})$.

**Proof.** Let $K_1$ and $K_2$ be the kernels in the integral representation of $u$ and $v$ in any case a) or b) according to Lemmas 5.1 and 5.2 respectively. Then the system (6.1) has a solution $(\Phi, A_\theta, A_z) \in C^2[0,R]^3$ if and only if $(\Phi, A_\theta, A_z) \in C[0,R]^3$ and for $0 \leq r \leq R$

$$
\Phi(r) = - \int_0^R K_1(r, s) h_\varphi(s, \Phi(s), A_\theta(s), A_z(s)) \, ds,
$$

$$
A_\theta(r) = - \int_0^R K_2(r, s) g_\varphi(s, \Phi(s), A_\theta(s), A_z(s)) \, ds,
$$

$$
A_z(r) = - \int_0^R K_1(r, s) k_\varphi(s, \Phi(s), A_\theta(s), A_z(s)) \, ds.
$$

Lemma 6.1 yields the a priori estimates,

$$
|\Phi(r)| \leq 16 \pi^2 \| m \|_1 \int_0^R K_1(r, s) \, ds,
$$

$$
|A_\theta(r)| \leq 8 \pi^2 \| m \|_1 \int_0^R K_2(r, s) \, ds,
$$

$$
|A_z(r)| \leq 8 \pi^2 \| m \|_1 \int_0^R K_1(r, s) \, ds.
$$

Because of the continuity of the kernels $K_1, K_2$ on $[0,R]^2$ and the Lipschitz-continuity of $h_\varphi, g_\varphi, k_\varphi$, we may apply Schauder's fixed point theorem and get a solution of (6.1). The regularity of $E, B$ follows from (2.22),(2.26),

$$
B'(r) = (A'_\theta(r) + \frac{A_\theta(r)}{r})' = A''_\theta(r) + \frac{A'_\theta(r)}{r} - \frac{A_\theta(r)}{r^2}
$$

$$
= g_\varphi(r, \Phi(r), A_\theta(r), A_z(r)),
$$
and \( g_\varphi(0, \xi, \eta, \zeta) = 0 \).

§7. Distribution Functions Depending Upon \( \mathcal{E} \) and \( \mathcal{F} \) (Singular Case)

We shall now investigate the system (2.24) in the case that the \( z \)-axis \( Z \) intersects \( \Omega \). For the sake of simplicity we assume that \( \Omega \) is the ball \( \mathcal{B} = \{ x \in \mathbb{R}^3 : |x| < R \} \). We shall first consider regularized problems by replacing the singular term \( \frac{1}{r^2} \) by \( \frac{1}{r^2_\delta} \), where \( r_\delta(x) := r_\delta(\sqrt{x_1^2 + x_2^2}) \), \( x \in \mathbb{R}^2 \) (see (5.3)), and then we let \( \delta \to 0 \). For the regularization, the restriction to \( \Omega = \mathcal{B} \) is not necessary.

**Lemma 7.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a cylindrically symmetric bounded domain with \( \partial \Omega \subset C^{2+\mu} \) for some \( \mu \in (0,1) \). Let \( \varphi \in C^1([1, \infty) \times \mathcal{B}) \) satisfy the assumptions of Lemma 4.1. Then for each sufficiently small \( \delta > 0 \) the problem

\[
-\Delta \Phi = h_\varphi(r, \Phi, A_\theta),
-\Delta A_\theta = g_\varphi(r, \Phi, A_\theta) - \frac{A_\theta}{r^2_\delta} \quad \text{in } \Omega,
\Phi = 0, \quad A_\theta = 0 \quad \text{on } \partial \Omega
\]

has a cylindrical symmetric solution \( \Phi_\delta, A_{\theta,\delta} \in C^{2+\mu}(\overline{\Omega}) \) (with a similar statement for \( (f, E, \mathcal{B}) \) as in Theorem 4.2). We have the uniform estimates

\[
0 \leq \Phi_\delta(x) \leq w_1(r(x)), \quad x \in \overline{\Omega} \setminus Z,
\tag{7.1}
\]

where \( w_1(r) := \frac{1}{4} \| h_\varphi \| (R^2 - r^2) \), \( W_\delta(r) := v_\delta(r) + C_\delta \| g_\varphi \| \delta \), and \( v_\delta \) is given by Lemma 5.4 for \( f(r) := -\| g_\varphi \| \) (\( C_\delta \) is the constant of Corollary 5.1).

**Proof.** We want to apply Theorem 4.1 and Corollary 4.1 and we need to construct sub- and supersolutions in \( C^1(\overline{\Omega}) \cap C^2(\Omega) \). We can use \( v_1 = 0 \) and \( w_1 \) as in the proof of Theorem 4.2 because \( w_1'(0) = 0 \) (see the remark preceding Theorem 6.1). From Corollary 5.1 we have \( W_\delta \geq 0 \) and \( W_\delta'(0) = v_\delta'(0) = c_\delta x_2'(0) + w_1'(0) = 0 \) by Lemma 5.3. For \( z_1 \geq 0 \) and \( z_2 := W_\delta(x) \),

\[
-\Delta W_\delta(x) = -\Delta v_\delta(x) = \| g_\varphi \| - \frac{v_\delta(x)}{r^2_\delta}
\geq g_\varphi(x, z_2) - \frac{W_\delta(x) - C_\delta \| g_\varphi \| \delta}{r^2_\delta}
\geq g_\varphi(x, z_2), \quad x \in \Omega.
\]

Hence \((0, -W_\delta)\) and \((w_1, W_\delta)\) are sub- and supersolutions.

By the uniqueness of the solution for Poisson's equation with right hand sides

\[
h_{\varphi, \delta}(x) := h_\varphi(r, \Phi_\delta(x), A_{\theta, \delta}(x)),
g_{\varphi, \delta}(x) := g_\varphi(r, \Phi_\delta(x), A_{\theta, \delta}(x))
\]
and with homogeneous boundary conditions we have for the ball $\mathcal{B}$

$$
\Phi_\delta(x) = \int_{\mathcal{B}} G(x,y) h_{\phi,\delta}(y) \, dy,
$$

(7.2)

$$
A_{\theta,\delta}(x) = \int_{\mathcal{B}} G(x,y) \left( g_{\phi,\delta}(y) - \frac{A_{\theta,\delta}(y)}{r_\delta^2(y)} \right) \, dy, \quad x \in \overline{\mathcal{B}},
$$

(7.3)

where $G$ is Green's function

$$
G(x,y) = \frac{1}{4\pi} \left( \frac{1}{|x-y|} - Q(x,y) \right)
$$

with

$$
Q(x,y) := \begin{cases} 
1/R & \text{for } x = 0, y \in \overline{\mathcal{B}}, \\
R & \frac{1}{|x|} \frac{1}{|x^* - y|} & \text{for } x \neq 0, y \in \mathcal{B} \cup (\partial \mathcal{B} \setminus \{x\}),
\end{cases}
$$

where $x^* := \frac{R^2}{x^T} x$ for $x \neq 0$. The following result is classical. If $f \in C(\overline{\mathcal{B}})$ then

$$
U(x) := \int_{\mathcal{B}} G(x,y) f(y) \, dy, \quad x \in \overline{\mathcal{B}}
$$

is an element of $C^1(\overline{\mathcal{B}})$ and

$$
|U(x)|, |DU(x)| \leq C_R \| f \|, \quad x \in \overline{\mathcal{B}};
$$

(7.4)

by $D, D^2, \cdots$ we denote partial derivatives of the respective order. If $f \in C^\alpha(\overline{\mathcal{B}})$ for some $0 < \alpha \leq 1$, then $U \in C^{2+\alpha}(\overline{\mathcal{B}})$, and

$$
\partial_{x_i} \partial_{x_j} U(x) = \int_{\mathcal{B}} (f(y) - f(x)) \partial_{x_i} \partial_{x_j} G(x,y) \, dy - \frac{1}{3} \delta_{ij} f(x)
$$

(7.5)

$$
|\partial_{x_i} \partial_{x_j} U(x)| \leq C_R (\| f \| + H_\alpha(f)), \quad x \in \overline{\mathcal{B}}
$$

$$
H_\alpha(\partial_{x_i} \partial_{x_j} U) \leq C_\alpha H_\alpha(f),
$$

where $H_\alpha(f)$ is the Hölder constant of $f$. This is the content of Müntz' Theorem, a direct proof of which has been given by S. Simoda [20]. In the present situation we can control the Hölder continuity of the derivatives of $f$ and $U$ only away from $Z$, and we shall have to refine Simoda's arguments. For $x, y \in \mathcal{B} \setminus \{0\}$ and for $x, y \in \partial \Omega$ with $x \neq y$ we have

$$
\frac{|x|}{R} |x^* - y| = \frac{|y|}{R} |x - y^*|.
$$

Hence for $x \in \mathcal{B}$

$$
Q(x,y) = \frac{R}{|y|} \frac{1}{|x - y^*|}, \quad y \in \overline{\mathcal{B}} \setminus \{0\}.
$$

Because $|x - y^*| \geq |x - y| \frac{R}{|y|}$, we have for $\lambda \geq 0$

$$
\frac{R}{|y|} \frac{1}{|x - y^*|^{1+\lambda}} \geq \frac{1}{|x - y|^{1+\lambda}}, \quad y \in \overline{\mathcal{B}}, 0 \neq y \neq x,
$$

and this implies

$$
|D^k_x G(x,y)| \leq \frac{C_k}{|x - y|^{1+k}}, \quad k = 0, 1, 2, 3.
$$

(7.6)
In the following we let $Z_{\eta} := \{ x \in \overline{B} : r(x) < \eta \}$, and for $f \in C^\alpha(\overline{B} \setminus Z_{\eta})$,

$$H_{\alpha, \eta}(f) := \sup\{ |f(y) - f(y')| : y, y' \in \overline{B} \setminus Z_{\eta}, |y - y'| < \alpha \},$$

$$H_{\alpha, \beta, \eta}(f) := \sup\{ |f(y) - f(y')| : y, y' \in \overline{B} \setminus Z_{\eta}, |y - y'| \leq \beta \},$$

$0 < \eta < R$, $0 < \alpha, \beta \leq 1$. For $x \in \mathbb{R}^3$, let $K_\alpha(x) := \{ y \in \mathbb{R}^3 : |y - x| < \alpha \}$.

**Lemma 7.2.** If $f \in C(\overline{B}) \cap C^\alpha(\overline{B} \setminus Z_{\eta/5})$, then $U \subseteq C^{2+\alpha}(\mathbb{R} \setminus Z_{\eta/5})$, and

$$|D^2U(x)| \leq C_{\alpha, R}(H_{\alpha, \eta/2}(f) + \| f \|_{L^\infty}(|\log \eta| + 1)), \quad x \in \overline{B} \setminus Z_{\eta},$$

(7.7)

$$H_{\alpha, 1/5, \eta}(D^2U) \leq C_{\alpha, R}(H_{\alpha, \eta/8}(f) + \| f \|_{L^\infty} \eta^{-1-\alpha}).$$

(7.8)

**Proof.** We still have the formula (7.5) for $x \in \overline{B} \setminus Z_{\eta}$. We estimate over $\overline{B} \cap K_{\eta/2}(x)$ and $\overline{B} \setminus K_{\eta/2}(x)$ separately ($y \in K_{\eta/2}(x)$ implies $r(y) \geq \eta/2$) and we get (7.7). For $p, q \in \overline{B} \setminus Z_{\eta}$ such that $0 < |p - q| \leq \eta/4$ and $K' := K_{[p-q](\mathbb{R}^2)} \subseteq \overline{B}$ (first case) one estimates

$$\int_{\mathbb{R} \setminus K'} (f(y) - f(x)) D^2G(x, y) dy \bigg|_{x = q} = \int_{\mathbb{R} \setminus K'} (f(y) - f(x)) D^2G(x, y) dy \bigg|_{x = p}$$

$$+ \int_{\mathbb{R} \setminus K'} (f(y) - f(p)) D^2G(x, y) dy \bigg|_{x = q} + (f(p) - f(q)) \int_{\mathbb{R} \setminus K'} D^2G(y, y) dy$$

to establish the inequality

$$H_{\alpha, 1/4, \eta}(D^2U) \leq C_{\alpha, R}(H_{\alpha, \eta/4}(f) + \| f \|_{L^\infty} \eta^{-1-\alpha}).$$

(7.9)

This is done similarly as in [20]: one uses $H_{\alpha, 5\eta/8}(f)$ in the first term, and with (7.6) one sees that the second term is bounded by

$$C_3 \int_0 \int_{|y - (p + sE)| \geq \frac{|E - s|}{2}} \frac{|f(y) + f(p + sE)|}{|p + sE - y|^4} dy ds,$$

$$+ C_3 \int_0 \int_{|y - (p + sE)| \geq \frac{|E - s|}{2}} \frac{|f(p + sE) - f(p)|}{|p + sE - y|^4} dy ds,$$

where $E := \frac{p - q}{|p - q|}$. The inner integral of the first of these two terms is estimated over the domains $\{ y \in \overline{B} : |y - (p + sE)| \geq \frac{|E - s|}{2}, r(y) \leq \eta/4 \}$ (where $|y - (p + sE)| \geq \eta/2$) and its complement, bounding the dominator by $2 \| f \|$ or by $H_{\alpha, \eta/4}(f)|y - (p + sE)|^\alpha$ respectively. In the second term $H_{\alpha, 3\eta/4}(f)$ is used. In the general case $p, q \in \overline{B} \setminus Z_{\eta}$ such that $0 < |p - q| \leq \eta/5$ one defines

$$p^{(0)} := \lambda p, \quad q^{(0)} := \lambda q \quad \text{with} \quad \lambda := \frac{|p| - |p - q|}{|p|},$$

and one sees that $p^{(0)}, q^{(0)}$ belongs to the first case with $\eta$ replaced by $4\eta/5$, and one can apply the argument with the chain of balls to prove the full assertion as in [20].

We still need a further result.
Lemma 7.3. a) If \( f \in C(\overline{B}) \), then
\[
V(x) := \int_{\overline{B}} G(x,y) \frac{f(y)}{r(y)} \, dy, \quad x \in \overline{B}
\]
is an element of \( C(\overline{B}) \cap C^1(\overline{B} \setminus \mathcal{Z}) \) and
\[
|V(x)| \leq C_R \| f \|, \quad x \in \overline{B}, \quad |DV(x)| \leq C_R \| f \| \| \log \eta \|, \quad x \in \overline{B} \setminus \mathcal{Z}. \tag{7.10}
b) If \( f \in C(\overline{B}) \cap C^\alpha(\overline{B} \setminus \mathcal{Z} \cap \mathcal{Z}) \), then \( V \in C^{2+\alpha}(\overline{B} \setminus \mathcal{Z}) \), and
\[
|D^2V(x)| \leq C_{R,\alpha}(H_{\alpha,\eta/2}(f) \eta^{-1} + \| f \| \eta^{-1-\alpha}),
\]
\[
H_{\alpha,1/5,\eta}(D^2V) \leq C_{R,\alpha}(H_{\alpha,\eta/5}(f) \eta^{-1} + \| f \| \eta^{-2-\alpha}). \tag{7.11}
\]

Proof. For \( \delta > 0 \) we define
\[
V_\delta(x) := \int_{\overline{B}} G(x,y) \frac{f(y)}{r_\delta(y)} \, dy, \quad x \in \overline{B}.
\]
By the remark preceding Lemma 7.2 we have \( V_\delta \in C^{2+\alpha}(\overline{B}) \),
\[
DV_\delta(x) = \int_{\overline{B}} DG(x,y) \frac{f(y)}{r_\delta(y)} \, dy,
\]
\[
\partial_{x_i} \partial_{x_j} V_\delta(x) = \int_{\overline{B}} \left( \frac{f(y)}{r_\delta(y)} - \frac{f(x)}{r_\delta(x)} \right) \partial_{x_i} \partial_{x_j} G(x,y) \, dy
\]
\[-\frac{1}{3} \delta_{ij} \frac{f(x)}{r_\delta(x)}, \quad x \in \overline{B}. \tag{7.12}
\]
It is easy to see that
\[
\int_{\overline{B}} \frac{1}{|x-y|^3} \frac{1}{r(y)} \, dy \leq \int_{\overline{B}} \frac{1}{|y|^3} \frac{1}{r(y)} \, dy < \infty, \quad x \in \overline{B},
\]
\[
\int_{\overline{B}} \frac{1}{|x-y|^2} \frac{1}{r(y)} \, dy \leq \int_{\overline{B}} \frac{1}{(|\eta,0,0|-y|^2} \frac{1}{r(y)} \, dy \leq C_R |\log \eta|, \quad x \in \overline{B} \setminus \mathcal{Z},
\]
\[
\int_{\overline{B}} \frac{1}{|x-y|^3-\alpha} \frac{1}{r(y)} \, dy \leq \int_{\overline{B}} \frac{1}{(|\eta,0,0|-y)^{3-\alpha}} \frac{1}{r(y)} \, dy \leq C_{R,\alpha} \eta^{\alpha-2}.
\]
Using (7.6), we get \( V_\delta \rightarrow V \) in \( C(\overline{B}) \) and in \( C^2(\overline{B} \setminus \mathcal{Z}) \), and (7.12) is true on \( \overline{B} \setminus \mathcal{Z} \) if we omit the index \( \delta \). For (7.11), we proceed as in Lemma 7.2, replacing \( f \) by \( f/r \). Because
\[
\frac{f(y)}{r(y)} - \frac{f(x)}{r(x)} = \frac{f(y) - f(x)}{r(y)} + f(x) \frac{r(x) - r(y)}{r(x)r(y)},
\]
we have
\[
H_{\alpha,\eta/5}(f) \leq H_{\alpha,\eta}(f) \eta^{-1} + C_R \| f \| \eta^{-1-\alpha}.
\]
Similarly as in (7.9) for \( p, q \in \overline{B} \setminus \mathcal{Z} \) such that \( 0 < |p - q| \leq \eta/4 \)
\[
C_3 \int_{\mathcal{Z}} \int_{\mathcal{Z} \setminus \mathcal{Z}} \left( \frac{\| f \|}{r(y)} + \frac{\| f \|}{r(p+sE)} \right) \frac{dy}{|p+sE-y|^2} ds.
\]
\begin{align*}
&\leq C_R |p-q|^\alpha \| f \| \eta^{-2-\alpha} + C_3 \frac{|p-q|}{\eta} \\
&\quad \int_0 ^{|z-(p+sE)| \geq \eta/2} \int_{\eta/4} ^r \| f \| \frac{dy}{|p+sE-y|^4} \ ds,
\end{align*}

and the second term is of the order of the first term.

In the sequel we let $C^{2+\alpha} _{loc}(\overline{B} \setminus \mathcal{Z}) := \bigcap _{\eta > 0} C^{2+\alpha}(\overline{B} \setminus \mathcal{Z})$ for $\alpha = 1$.

**Theorem 7.1.** Let $B := \{ x \in \mathbb{R}^3 : |x| < R \}$, $\mathcal{Z} := \{(0,0,x_3) : x_3 \in \mathbb{R} \}$. Let $\varphi \in C^1([1,\infty) \times \mathbb{R})$ satisfy the assumptions of Theorem 4.2. Then the problem

$$
-\Delta \Phi = h_\varphi(r,\Phi,\Phi_0),
$$

$$
-\Delta \Phi_0 = g_\varphi(r,\Phi,\Phi_0) - \frac{A_\Phi}{r^2} \quad \text{in } B \setminus \mathcal{Z},
$$

$$
\Phi = 0, \quad \Phi_0 = 0 \quad \text{on } \partial B
$$

has a cylindrically symmetric solution

$$
\Phi \in C^1(\overline{B}) \cap C^{2+1}_{loc}(\overline{B} \setminus \mathcal{Z}), \quad \Phi_0 \in C(\overline{B}) \cap C^{2+1}_{loc}(\overline{B} \setminus \mathcal{Z}).
$$

We have

$$
0 \leq \Phi(r,z) \leq \frac{1}{4} \| h_\varphi \| (R^2 - r^2),
$$

$$
|\Phi_0(r,z)| \leq \frac{1}{3} \| g_\varphi \| r(R-r). \quad (7.13)
$$

For the corresponding stationary solution such that $f = \varphi(E,F)$ we have $f \in C(\overline{B}) \cap C^1(\overline{B} \setminus \mathcal{Z})$, $E \in C(\overline{B})^3 \cap C^{1+1}_{loc}(\overline{B} \setminus \mathcal{Z})^3$, $B \in C^{1+1}_{loc}(\overline{B} \setminus \mathcal{Z})^3$ and $|B(r,z)| \leq C_{\varphi,R} |\log r|$

**Proof.** It follows from (7.2) and (7.4) that

$$
|D\Phi_\delta(x)| \leq C_R \| h_\varphi, \delta \| \leq C_R \| h_\varphi \|. \quad (7.14)
$$

We write (7.3) as

$$
A_{\varphi,\delta}(x) = \int_B G(x,y) \left( g_{\varphi,\delta}(y) - \frac{1}{r(y)} \left( r(y) \frac{A_{\varphi,\delta}(y)}{r_{\delta}(y)} \right) \right) \ dy
$$

and note that $r(y) \frac{A_{\varphi,\delta}(y)}{r_{\delta}(y)}$ is a continuous function on $\overline{B}$ which can be estimated by Lemma 7.1 and Corollary 5.1:

$$
|A_{\varphi,\delta}(x)| \leq W_\delta(r) = v_\delta(r) + C_R^\star \| g_\varphi \| \delta
$$

$$
\leq v_\delta(r) + 2C_R^\star \| g_\varphi \| \delta
$$

$$
= \| g_\varphi \| \left( \frac{1}{3} r(R-r) + 2C_R^\star \delta \right), \quad (7.15)
$$

such that

$$
\left| r \frac{A_{\varphi,\delta}(y)}{r_{\delta}(y)} \right| \leq \| g_\varphi \| \left( \frac{1}{3} r^2 (R-r) + 2C_R^\star r_{\delta} \right) \leq C_R \| g_\varphi \|. \quad (7.16)
$$

Hence we obtain from Lemma 7.3

$$
|DA_{\varphi,\delta}(x)| \leq C_R \| g_\varphi \| \| \log \eta \|, \quad x \in \overline{B} \setminus \mathcal{Z}. \quad (7.16)
$$

Theorem 4.2, the estimates (7.14), (7.15), (7.16) imply with the chainrule

$$
|Dh_{\varphi,\delta}(x)|, |Dg_{\varphi,\delta}(x)| \leq C_{\varphi,R,\eta}, \quad x \in \overline{B} \setminus \mathcal{Z}. \quad (7.16)
$$
Lemmas 7.2 and 7.3 then imply that \( \{\Phi_\delta\} \) is bounded in \( C^1(\mathbb{B}) \) and in \( C^{2+\eta}(\mathbb{B}\setminus Z_\eta) \) for each \( \eta > 0 \). By compactness, there exists a sequence \( \delta_n \downarrow 0 \) and functions \( \Phi \in C(\mathbb{B}) \cap C^{2+\eta}_{loc}(\mathbb{B}\setminus Z) \), \( A_\delta \in C^{2+\eta}_{loc}(\mathbb{B}\setminus Z) \) such that \( \Phi_{\delta_n} \to \Phi \) in \( C(\mathbb{B}) \) and \( C^{2+\eta}(\mathbb{B}\setminus Z_\eta) \) and \( A_{\delta_n} \to A_\delta \) in \( C^{2+\eta}(\mathbb{B}\setminus Z_\eta) \) for each \( \eta > 0 \). If we define \( A_\delta(x) = 0 \) for \( x \in \mathbb{B}\cap Z \), then also \( A_{\delta_n} \to A_\delta \) in \( C(\mathbb{B}) \) and (7.13) is valid. This follows from (7.15). The limits satisfy

\[
\Phi(x) = \int_{\mathbb{B}} G(x,y) h_\phi(y,\Phi(y),A_\delta(y)) \, dy,
\]
\[
A_\delta(x) = \int_{\mathbb{B}} G(x,y) \left( g_\phi(y,\Phi(y),A_\delta(y)) - \frac{A_\delta(y)}{r^2(y)} \right) \, dy, \quad x \in \mathbb{B}
\]

and hence \( \Phi, A_\delta \) are solutions of the problem. The asserted regularity of \( f \) and \( E \) is obvious.

Using (2.22), we get in \( \mathbb{B}\setminus Z \)

\[
B_r = -\partial_x A_\delta, \quad B_\theta = 0, \quad B_z = \frac{1}{r} A_\delta + \partial_x A_\delta.
\]

The logarithmic estimate of \( B \) follows from (7.13) and (7.16).

In Theorem 7.1 it might not be excluded that \( B \) exists as a continuous function up to the axis \( Z \). To prove the existence of a Green's function for the operator \(-\Delta + \frac{1}{r^2}\) and the generalization to arbitrary cylindrically symmetric domains \( \Omega \) seems to be a topic of further research.

Acknowledgement. The first author wants to thank Professor Li Ta-tsien, Fudan University, for the hospitality during his visit in September 1992.

References


