IRRATIONAL ROTATION $C^*$-ALGEBRA
FOR GROUPOID $C^*$-ALGEBRA**

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Abstract

This paper characterizes the irrational rotation $C^*$-algebra associated with the Toeplitz $C^*$-algebra over the $L$-shaped domain in $D^2$ in the sense of the maximal radical series, which is an isomorphism invariant.

Keywords $L$-shaped domain, Toeplitz $C^*$-algebra, Irrational rotation, Invariant subset, Maximal ideal.

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§1. Preliminaries

Irrational rotation $C^*$-algebra on the unit circle was first studied by M. Rieffel in [13]. Since 1980's many people have paid great attention to this subject (see [15], [16], [17], [18] and [19]). It has played an important part in the analysis of $C^*$-algebras, $K$-theory and index theory. In recent years the study of rotation $C^*$-algebras on group $C^*$-algebras and Toeplitz $C^*$-algebras has been developed. For Example, in [19] Handelman and Yin obtained a complete invariant for rotation $C^*$-algebra of Toeplitz $C^*$-algebra on the polydisk. In this paper from the view of groupoid we establish the structure of rotation $C^*$-algebras of Toeplitz $C^*$-algebras on $L$-shaped domain in $D^2$. This idea will get further developing in our other papers.

Suppose that $Y$ is a locally compact Hausdorff and second countable space, and $X$ is a both open and compact subset of $Y$. $\mathbb{Z}^n$ acts on $Y$ on the right continuously so that $(Y \mathbb{Z}^n)$ is a transformation group. For the $n$-tuple $\theta$ in $\mathbb{Z}^n$, define the homomorphism $C_\theta : Y \times \mathbb{Z}^n \to \mathbb{I}$ by

$$C_\theta(y, p) = \theta^p.$$  

Denote the reduction of $Y \times \mathbb{Z}^n$ on $X$ by $G$. Then the reduction of the skew product $(Y \times \mathbb{Z}^n)(C_\theta)$ on $X \times \mathbb{I}$ is the skew product $G(C_\theta)$.

**Proposition 1.1.** The groupoid $G$ and $G(C_\theta)$ are r-discrete and amenable.

The skew product $G(C_\theta) = G \times_{C_\theta} \mathbb{I}$ is a locally compact groupoid with composable pairs

$$G(C_\theta)^{(2)} = \{ ((x, p, a), (y, q, b)) | ((x, p), (y, q)) \in G^{(2)} \text{ and } b = a\theta^p \}.$$
The product is 
\[(x, p, a)(y, q, b) = (x, p + q, a),\]
and the inverse is 
\[(x, p, a)^{-1} = (x + p, -p, a\theta^p).\]
The domain map is 
\[d(x, p, a) = (x + p, 0, \theta^p),\]
and the range map is 
\[r(x, p, a) = (x, 0, a).\]
So the unit space may be identified with \(X\) and the range map is \(T\), named \(\times\).

**Remark 1.1.** Indeed, \(G(C_\theta)\) is the reduction of the skew product of \((Y \times \mathbb{Z}^p)(C_\theta)\) on \(X \times \mathbb{I}^n\).

**Proposition 1.2.** \(C^*(G(C_\theta)) \cong C^*(G) \times_{\alpha_\theta} \mathbb{Z}^p\).**

**Proposition 1.3.** The groupoid \((Y \times \mathbb{Z}^p)(C_\theta)\) is principal if there is no solution of nonzero integers to the equation \(\theta^p = 1\).

**Proof.** We have to prove that the isotropy group \((Y \times \mathbb{Z}^n)(C_\theta)]_u\) at every point \(u\) in the unit space is trivial. Suppose that \((x, m, a)\) is in the isotropy group of \(u\). Then we have
\[(x + p, 0, \theta^p a) = (x, 0, a) = u.\]
It follows that \(\theta^p = 1\). Therefore \(p = 0\) and \((x, m, a) = u.\)

**Remark 1.2.** \(G(C_\theta)\) is principal if \((Y \times \mathbb{Z}^n)(C_\theta)\) is.

It is easy to prove the following lemma.

**Lemma 1.1.** Suppose that \(G\) is a groupoid and \(A\) a group. Let \(c : G \to A\) be a homomorphism. \(G(c)\) is the skew product \(G \times_c A\). Suppose that \(E\) is a subset of \(G^0\). Then \(E \times A\) is invariant in \(G(c)\) if \(E\) is invariant in \(G\).

Let \(\Omega = \{(z_1, z_2) \in \mathbb{D}^2 | |z_1| < \delta_1, |z_2| < 1 \text{ or } |z_1| < 1, |z_2| < \delta_2\}\), where \(\delta_1, \delta_2 < 1\). Then \(\Omega\), named \(L\)-shaped domain, is a Reinhardt domain in \(\mathbb{D}^2\). P. E. Curto and P. S. Muhly have represented the Toeplitz \(C^*-\)algebra \(C^*(\Omega)\) faithfully by a groupoid \(C^*-\)algebra \(C^*(\mathbb{D})[\mathbb{T}]\). Let us repeat the procedure briefly here with some new notations introduced. Let \(T(p) : A^2(\Omega) \to A^2(\Omega)\) be the Toeplitz operator of the symbol \(z^p\). Then \(\{T(p) | p \in \mathbb{Z}^+\}\) is a contractable representation of \(\mathbb{Z}^+\) by a weighted function
\[w_+(p, q) = \|z^{p+q}\|,\]
for \(p, q \in \mathbb{Z}^+\).

A direct calculation shows
\[w_+(\epsilon_1, p) = \sqrt{(p_1 + 1)(\delta_1^{2p_1 + 4} + \delta_2^{2p_2 + 2} - \delta_1^{2p_1 + 4} \delta_2^{2p_2 + 2})/(p_1 + 2)(\delta_1^{2p_1 + 4} + \delta_2^{2p_2 + 2} - \delta_1^{2p_1 + 4} \delta_2^{2p_2 + 2})},\]
and
\[w_+(\epsilon_2, p) = \sqrt{(p_2 + 1)(\delta_1^{2p_1 + 2} + \delta_2^{2p_2 + 4} - \delta_1^{2p_1 + 2} \delta_2^{2p_2 + 4})/(p_2 + 2)(\delta_1^{2p_1 + 2} + \delta_2^{2p_2 + 2} - \delta_1^{2p_1 + 2} \delta_2^{2p_2 + 2})}.\]
Extend each \( w_+ (p, \cdot) \) to \( \mathbb{Z}^2 \) by taking zero on \( \mathbb{Z}^2 \setminus \mathbb{Z}_+^2 \). Let \( A \) denote the translation-invariant \( C^* \)-subalgebra of \( \ell^\infty (\mathbb{Z}^2) \) generated by the family \( \{ w(p, \cdot) \mid p \in \mathbb{Z}_+ \} \) not including the identity. The maximal ideals space of \( A \), denoted by \( Y \), is locally compact and second countable. The natural action, \( \tau : \mathbb{Z}^2 \to \text{Aut}(A) \), defined by translation induces an action of \( \mathbb{Z}^2 \) on \( Y \) according to this prescription: \( (y + p)(a) = y(\tau_p(a)) \). Since the evaluation at \( p \) gives a multiplicative linear functional, say \( \alpha(p) \), we get an injection \( \alpha : \mathbb{Z}^2 \to Y \) both open and continuous. The subset \( \bar{\alpha}(\mathbb{Z}^2) \), denoted by \( X \), is open and compact. \( G \) is the reduction of \( Y \times \mathbb{Z}^2 \) by \( X \) as defined above. Then \( C^*(\Omega) \) is faithfully represented by \( C^*(G) \) (see [7]).

According to [7], \( Y \) consists of four parts, i.e.,

\[
Y = \alpha(\mathbb{Z}^2) \cup \alpha(\mathbb{Z} \times \{ \infty \}) \cup \alpha(\{ \infty \} \times \mathbb{Z}) \cup \beta([-\infty, +\infty]) ,
\]

where

\[
\alpha(p_1, \infty) = \lim_{p_2 \to +\infty} \alpha(p_1, p_2).
\]

and

\[
\alpha(\infty, p_2) = \lim_{p_1 \to +\infty} \alpha(p_1, p_2)
\]

in \( Y \); and \( \beta : [-\infty, +\infty] \to \infty \) is the realization of the subset, \( \infty \), of \( Y \) consisting of all the possible limits \( \lim_{k_1,k_2 \to +\infty} \alpha(k) \) in \( Y \). Indeed, \( \beta(t) \) is uniquely determined by

\[
(\beta(t)(w(\epsilon_1, \cdot)), \beta(t)(w(\epsilon_2, \cdot))) = \begin{cases} (\delta_1, 1) & \text{for } t = -\infty, \\ \left( \frac{\delta_1^2 + \exp(t)}{1 + \exp(t)}, \sqrt{\frac{1 + \delta_2^2 \exp(t)}{1 + \exp(t)}} \right) & \text{for } t \in \mathbb{R}, \\ (1, \delta_2) & \text{for } t = +\infty. \end{cases}
\]

Thus

\[
X = \alpha(\mathbb{Z}_+^2) \cup \alpha(\mathbb{Z}_+ \times \{ \infty \}) \cup \alpha(\{ \infty \} \times \mathbb{Z}_+) \cup \beta([-\infty, +\infty]).
\]

Given a pair of numbers \( \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 \), satisfying the condition that there is no nonzero integer \( n \) such that \( \theta_1^n = 1 \) or \( \theta_2^n = 1 \), which is weaker than that in Proposition 1.3, there is an automorphism \( \varphi_\theta : \Omega \to \Omega \) defined via

\[
\varphi_\theta(z_1, z_2) = (\theta_1 z_1, \theta_2 z_2), \text{ for } (z_1, z_2) \in \Omega.
\]

Thus there is an induced \( C^* \)-dynamical system \( (C^*(\Omega), \mathcal{Z}, \varphi_\theta) \), where \( \varphi_\theta \) is the induced automorphism of \( C^*(\Omega) \) such that \( \varphi_\theta(T_f) = T_{f \circ \varphi_\theta^{-1}} \) for \( f \in C(\Omega) \).

**Proposition 1.4.** \( C^*(G) \times_{\alpha_{\theta}} \mathcal{Z} \cong C^*(\Omega) \times_{\varphi_\theta} \mathcal{Z} \).

**Remark 1.3.** \( \lim_{p_1 \to +\infty} \alpha(p_1, +\infty) = \beta(-\infty) \) and \( \lim_{p_2 \to +\infty} \alpha(+\infty, p_2) = \beta(+\infty) \).

§2. Invariant Maximal Radical Series of \( C^*(G(\theta)) \)

The maximal radical series of a \( C^* \)-algebra is invariant under the isomorphism. It plays an important part in the classification of some \( C^* \)-algebras. By the definition[20], the maximal radical of a \( C^* \)-algebra \( A \) is the intersection of all closed two-sided maximal ideals of \( A \), and is denoted by \( m(A) \), the composition series

\[
\cdots \triangleleft m(m(A)) \triangleleft m(A) \triangleleft A.
\]
is called the maximal radical series. In this section we will determine the maximal radical series of the rotational $C^*$-algebra $C^*(G(C_0))$.

By [1], there is an order-preserving homomorphism from the family of the invariant open subsets to the family of the closed ideals in the reduced groupoid $C^*$-algebra. And now, we will first determine the minimal invariant closed subsets in the groupoid $G(C_0)$.

**Lemma 2.1.** There are only two minimal invariant closed subsets in the unit space of the groupoid $G(C_0)$, i.e., $\{\beta(+)\} \times \mathbb{I}$ and $\{\beta(-)\} \times \mathbb{I}$, denoted by $F_1$ and $F_2$ respectively. Their complements are denoted by $B_1$ and $B_2$ respectively. Any invariant closed subset contains at least one of the $F_i$’s.

**Proof.** The $F_i$’s are obviously minimal invariant and closed. Given an invariant closed subset $F$, take any $u \in F$.

1) If $u$ is in either $F_1$ or $F_2$, then $F_i \subseteq F$ or $F_2 \subseteq F$.

2) If $u$ is in $\beta(B) \times \mathbb{I}$, then

$$\lim_{m \to +\infty} (u + (0, m)) = (\beta(+)t),$$

for some $t$ in $\mathbb{I}$. Hence $F \cap F_1 \neq \emptyset$, and by 1) $F_1 \subseteq F$.

3) If $u$ is in $\alpha(\mathbb{Z} \times \{\infty\}) \times \mathbb{I}$, then

$$\lim_{m \to +\infty} (u + (m, 0)) = (\beta(-)t),$$

for some $t$ in $\mathbb{I}$. Hence $F \cap F_2 \neq \emptyset$, and by 1) $F_2 \subseteq F$.

4) If $u$ is in $\alpha(\{\infty\} \times \mathbb{Z}^+) \times \mathbb{I}$, then by the same reason as above, $F_1 \subseteq F$.

5) If $u$ is in $\alpha(\mathbb{Z}^+ \times \{\infty\}) \times \mathbb{I}$, then

$$\lim_{m \to +\infty} (u + (m, 0)) = (\alpha(\infty, n)t),$$

for some $t$ in $\mathbb{I}$. Hence by the same reason as in 4), $F_1 \subseteq F$.

The lemma follows now.

**Remark 2.1.** We have used the fact that $\{\theta^p | p \in \mathbb{Z}^+\}$ is dense in $\mathbb{I}$ if there is no integer $n$ of nonzero such that $\theta^n_1 = 1$ or $\theta^n_2 = 1$.

**Lemma 2.2.** If the ratio $\frac{\ln \delta_1}{\ln \delta_2}$ is irrational, the isotropy group $G(C_0)|u$ is trivial for $u \notin F_1 \cup F_2$, i.e., $u \in B$.

**Proof.** For any $(x, p, t)$ in $G(C_0)|u$, we have two equalities

$$x + p = x, \quad (I)$$

$$\theta^p = 1. \quad (II)$$

1) If $x$ is in $\alpha(\mathbb{Z}^+ \times \{\infty\})$, say $x = \alpha(q)$, then equality (I) becomes $\alpha(q+p) = \alpha(q)$. Consequently, $p = 0$ since $\alpha$ is injective.

2) If $x$ is in $\alpha(\mathbb{Z} \times \{\infty\})$, say $x = \alpha(m, +\infty)$, then equality (I) becomes $\alpha(m + p_1, +\infty) = \alpha(m, +\infty)$. Thus $p_1 = 0$. It follows that $p_2 = 0$ from equality (II).

3) If $x$ is in $\alpha(\{\infty\} \times \mathbb{Z}^+)$, then $p = 0$ by the same reason as in case (2).

4) If $x$ is in $\beta(-\infty, +\infty)$, say $x = \beta(s)$, then equation (I) becomes $\beta(s + 2p_2 \ln \delta_2 - 2p_1 \ln \delta_1) = \beta(s)$. It follows that $p_2 \ln \delta_2 = p_1 \ln \delta_1$. Therefore $p = 0$.

Finally we get $(x, p, t) = u$. The lemma follows.

**Theorem 2.1.** The maximal radical of the groupoid $C^*$-algebra $C^*(G(C_0))$ is $I(B)$. 

Proof. If 1, \( \frac{\arg \theta_1}{2\pi} \) and \( \frac{\arg \theta_2}{2\pi} \) are linearly independent over the field \( \mathcal{Q} \) of the rational numbers, the groupoid is principal, the maximal closed ideals are \( I(B_1) \) and \( I(B_2) \). Therefore the intersection of the maximal closed ideals is \( I(B) \).

If 1, \( \frac{\arg \theta_1}{2\pi} \) and \( \frac{\arg \theta_2}{2\pi} \) are linearly dependent over the field \( \mathcal{Q} \), we will prove the following claims:

1. There is indeed a maximal closed ideal in the groupoid \( C^* \)-algebra and the intersection of the maximal closed ideals is contained in \( I(B) \).

2. Each maximal closed ideal \( I \) in the groupoid \( C^* \)-algebra contains \( I(B) \).

And now

\[
C^*(G(C_\theta))/I(B_1) \cong C^*(G(C_\theta)|_{F_1}) = C^*(\beta\{+\infty\} \times \mathbb{F} \times \mathbb{Z}^2) \\
\cong C^*(\mathbb{F} \times \mathbb{Z}^2) \\
\cong C(\mathbb{F}^2) \times_{\alpha_\theta} \mathbb{Z}
\]

where \( \alpha_\theta(f)(\lambda_1, \lambda_2) = f(\theta_1\lambda_1, \theta_2\lambda_2) = f \cdot \varphi_\theta(\lambda_1, \lambda_2) \). Since the homeomorphism \( \varphi_\theta \) is not minimal, the crossed product \( C(\mathbb{F}^2) \times_{\alpha_\theta} \mathbb{Z} \) is not simple by [1]. However, the nontrivial closed ideal must be contained in some maximal ones since the crossed product is unital. Suppose that \( I \) is the maximal closed ideal in the crossed product. Then the quotient \( C(\mathbb{F}^2) \times_{\alpha_\theta} \mathbb{Z}/I \) is simple. So there is a surjective homomorphism from \( C^*(G(C_\theta)) \) onto \( C(\mathbb{F}^2) \times_{\alpha_\theta} \mathbb{Z}/I \), whose kernel is a maximal closed ideal in \( C^*(G(C_\theta)) \).

Since the closed orbit \( \{\varphi_\theta^n(\lambda) | n \in \mathbb{Z}^2 \} \) is minimal for every \( \lambda \in \mathbb{F}^2 \), by [1] the intersection of the maximal closed ideals in the crossed product is \( \{0\} \). Hence the maximal radical is contained in \( I(B_1) \). A similar argument shows that the maximal radical is contained in \( I(B_2) \). The claim (1) follows.

For each maximal closed ideal \( I \), there is an integrated representation, \( \pi = (\mu, L, \mathcal{H}) \), of \( C^*(G(C_\theta)) \) with kernel \( I \). It follows that the representation \( \pi \) is weakly contained in the induced left regular representation living on \( \mu \). Therefore \( I(F) \subseteq \ker(\pi) \), where \( F \) denotes the support of \( \mu \). \( F \) is an invariant closed subset. By the proof of Lemma 2.1, \( F \) contains either \( F_1 \) or \( F_2 \).

If \( F = F_1 \) or \( F = F_2 \), then \( I(B_1) \subseteq I \) or \( I(B_2) \subseteq I \); thus \( I(B) \subseteq I \).

If \( F \neq F_1 \) and \( F \neq F_2 \), then

1) \( F \) only contains \( F_1 \). Then \( \bar{F} \setminus F_1 \) is a nontrivial invariant closed subset, say \( \bar{F}_2 \), contained in \( F \). Therefore it contains \( F_1 \), i.e., \( \bar{F}_2 = F \). By the proof of Proposition 4.4 in [1],

\[
\sup_{u \in F} |f(u)| \leq \|\pi(f)\|
\]

it follows that \( I \subseteq I(B_1) \). Therefore \( I = I(B_1) \). Thus \( I(B) \subseteq I \).

2) \( F \) only contains \( F_2 \). Then \( I(B) \subseteq I \) by the same reason as in case 1.

3) \( F \) contains both \( F_1 \) and \( F_2 \). Set

\[
F' = F \setminus (F_1 \cup F_2), \quad \mu'(E) = \mu(E \cap F'), \quad \mu_1(E) = \mu(E \cap (F_1 \cup F_2)).
\]

Then \( \pi_1 = (\mu_1, L, \mathcal{H}) \) and \( \pi' = (\mu', L, \mathcal{H}) \) are the integrated representations of the groupoid.
$C^*$-algebra $C^*(G(C_0))$. Moreover we have
\[ \pi_1(f)(\zeta) = \pi(f)(\chi_{F_1 \cup F_2} \xi) = \chi_{F_1 \cup F_2} \pi(f)(\zeta), \]
\[ \pi'(f)(\zeta) = \pi(f)(\chi_F \xi) = \chi_F \pi(f)(\zeta), \]
\[ \ker(\pi) = \ker(\pi_1) \cap \ker(\pi'). \]
Since $\ker(\pi)$ is a maximal closed ideal, it coincides with either $\ker(\pi_1)$ or $\ker(\pi')$.

(1) If $I = \ker(\pi_1)$, it follows immediately that $I(B) \subseteq I$.
(2) If $I = \ker(\pi')$, one of the following cases occurs.
(i) $\text{supp}(\mu') = \overline{F}$ contains $F_1 \cup F_2$. By the proof of Proposition 4.4 in [1] we get
\[ \sup_{u \in F} |f(u)| \leq \|\pi(f)\|, \]
and it follows immediately that $I = I(G(C_0)^0 \setminus \overline{F})$. Since $B_1$ and $B_2$ are the maximal invariant open subsets and
\[ I(G(C_0)^0 \setminus \overline{F}) \subseteq I(B_1) \cap I(B_2), \]
this case can not occur.
(ii) $\text{supp}(\mu')$ contains only one of the $F_i$'s. Then by the above discussion we have $I(B) \subseteq I$.

The claim (2) follows now. The theorem follows from the above claims.

**Lemma 2.3.** The groupoid $G(C_0)|_B$, denoted by $G(C_0)'$, is $r$-discrete, principal and amenable.

**Lemma 2.4.** The maximal radical of $C^*(G(C_0))$ is $I(\alpha(Z_+^2) \times \mathbb{I})$, denoted by $C^*(G(C_0))^{\prime\prime}$.

**Proof.** By Lemma 2.2 and [1], there is an order-preserving isomorphism between the family of the maximal closed ideals in $C^*(G(C_0))$ and the family $\mathfrak{I}$ of the maximal invariant open subsets in $G(C_0)$. Let $\bigcap_{B \in \mathfrak{I}} I(B) = I$. Then there is an invariant open subset $\widehat{B}$ such that $I = I(\widehat{B})$. We find that $\widehat{B} = \text{int} \bigcap_{B \in \mathfrak{I}} B$. Let us determine the minimal invariant closed subsets in the unit space of the groupoid $G(C_0)'$. Note first that any minimal invariant closed subset of the unit space must be a closed orbit $[\mathfrak{I}]$ for some $t$ in the unit space.

The unit space of $G(C_0)'$ consists of four disjoint parts,
\[ \alpha(Z_+^2) \times \mathbb{I}, \ \alpha(Z_+ \times \{\infty\}) \times \mathbb{I}, \ \alpha(\{\infty\} \times Z_+) \times \mathbb{I} \]
and $\beta(R) \times \mathbb{I}$. The first part is an invariant open subset, while the last ones are invariant closed subsets.

Given $u$ in the unit space, we proceed in the following four cases.
(1) $u \in \beta(R) \times \mathbb{I}$, say $u = (\beta(s), t)$. Define the distance function $d$ on $\beta(R) \times \mathbb{I}$ by
\[ d((\beta(s), t), (\beta(s'), t')) = |s - s'| + |t - t'|. \]
Then the distance is an invariance under the action of $Z^2$ on $\beta(R) \times \mathbb{I}$. For each $v \in [u]$ there is a sequence $\{p_m\}_{m=1}^\infty$ in $Z^2$ such that $v = \lim_{m \to \infty} (u + p_m)$. However
\[ \lim_{m \to \infty} d(u, v - p_m) = \lim d(u + p_m, v) = 0. \]
It follows that the closed orbits in $\beta(R) \times \mathbb{I}$ are either disjoint or identical. So the closed orbits in $\beta(R) \times \mathbb{I}$ are the minimal invariant closed subsets in the unit space.
(2) $u \in \alpha(Z_+ \times \{\infty\}) \times \mathbb{I}$, say $u = (\alpha(n, \infty), t)$. Now the subset
\[ S := \{u + (0, m) = (\alpha(n, \infty), t\theta^m_2)|m \in \mathbb{Z}\} \]
is contained in the orbit \([u]\). It follows that \(\{\alpha(n, \infty)\} \times \mathbb{I}\) is contained in the closed orbit \([u]\). For any \(k \in \mathbb{Z}_+\)

\[u + (k - n, 0) = (\alpha(k, \infty), t\theta_1^{k-n})\]

It follows that

\[\alpha(\mathbb{Z}_+ \times \{\infty\}) \times \mathbb{I} = [u].\]

Therefore \(\alpha(\mathbb{Z}_+ \times \{\infty\}) \times \mathbb{I}\) is a minimal invariant closed subset in the unit space.

(3) \(u \in \alpha(\{\infty\} \times \mathbb{Z}_+) \times \mathbb{I}\). By the same reason as that in case (2), \(\alpha(\{\infty\} \times \mathbb{Z}_+) \times \mathbb{I}\) is a minimal invariant closed subset in the unit space.

(4) \(u \in \alpha(\mathbb{Z}_+^2) \times \mathbb{I}\), say \(u = (\alpha(p), t)\). Now the closed orbit \([u]\) contains at least one point in \(\alpha(\mathbb{Z}_+ \times \{\infty\}) \times \mathbb{I}\) and therefore contains \(\alpha(\mathbb{Z}_+ \times \{\infty\}) \times \mathbb{I}\), so the closed orbit \([u]\) is not minimal.

So the family of the minimal invariant closed subsets in the unit space is

\[\{\alpha(\mathbb{Z}_+ \times \{\infty\}) \times \mathbb{I}; \alpha(\{\infty\} \times \mathbb{Z}_+) \times \mathbb{I}; [u]u \in \beta(\mathbb{R}) \times \mathbb{I}\},\]

whose union is

\[\alpha(\mathbb{Z}_+ \times \{\infty\}) \times \mathbb{I} \cup \alpha(\{\infty\} \times \mathbb{Z}_+) \times \mathbb{I} \cup \beta(\mathbb{R}) \times \mathbb{I}.\]

Therefore the intersection of the maximal invariant open subsets in the unit space is \(\alpha(\mathbb{Z}_+^2) \times \mathbb{I}\). The lemma follows now.

**Lemma 2.5.** The intersection of the maximal invariant open subsets in \(G(C_\theta)|_{\alpha(\mathbb{Z}_+^2) \times \mathbb{I}}\) is empty. Consequently the maximal radical of \(C^*(G(C_\theta))^{\gamma}\) is zero.

**Proof.** Given \(u \in \alpha(\mathbb{Z}_+^2) \times \mathbb{I}\) the closed orbit created by \(u\) is exactly the orbit created by \(u\). So every orbit in the unit space \(\alpha(\mathbb{Z}_+^2) \times \mathbb{I}\) is a minimal invariant closed subset. The lemma follows now.

In summary, we obtain the maximal radical series,

\[0 < C^*(G(C_\theta)^{\gamma}) < C^*(G(C_\theta)^\theta) < C^*(G(C_\theta)),\]

for the groupoid \(C^*\)-algebra \(C^*(G(C_\theta))\) in the case that both \(\frac{\arg \theta_1}{2\pi}\) and \(\frac{\arg \theta_2}{2\pi}\) are irrational. It is invariant under the isomorphism.

**Remark 2.2.** The classification and the \(K\)-theory of the rotational \(C^*\)-algebras will be given in our following paper.

**References**


