DIMENSIONS OF SELF-AFFINE SETS WITH OVERLAPS**

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Abstract

The authors develop an algorithm to show that a class of self-affine sets with overlaps can be viewed as sofic affine-invariant sets without overlaps, thus by using the results of [11] and [10], the Hausdorff and Minkowski dimensions are determined.

Keywords Self-affine set, Overlap, Hausdorff dimension, Minkowski dimension

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§1. Introduction

1.1. McMullen’s Self-Affine Carpets

McMullen[11] and Bedford[1] independently studied plane sets constructed as follows. Let $1 < m \leq n$ be integers. By drawing $n - 1$ vertical lines and $m - 1$ horizontal lines, partition the unit square into $nm$ congruent rectangles. Let $S$ be a subcollection of these rectangles; erase all other rectangles and partition the remaining ones into $nm$ congruent subrectangles, again keeping only those which correspond to the pattern $S$. Repeating this procedure and infinitum, a compact set $K$ is obtained, which is called McMullen’s self-affine carpet.

Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. The McMullen’s self-affine carpets can be viewed as subsets of $T^2$. They are invariant under the toral endomorphism

$$T = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}.$$ 

Denote by $D_0 = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$. Let $D \subset D_0$. Then the corresponding self-affine carpets are

$$K(T, D) = \left\{ \sum_{k=1}^{\infty} T^{-k}d_k : d_k \in D \right\}.$$

The set $K(T, D)$ can be also viewed as the attractor for the family of $|D|$ affine contractions

$$f_j \left( \begin{array}{c} x \\ y \end{array} \right) = T^{-1} \left( \begin{array}{c} x \\ y \end{array} \right) + d_j,$$

where $d_j \in D$, i.e., $K(T, D)$ is the unique compact invariant set of the above iterated function system[8].
Though these sets are the simplest self-affine sets, when \( n > m \) the different expansion coefficients in the horizontal and vertical directions make the quantitative analysis of these sets quite delicate; McMullen and Bedford succeeded in calculating their Hausdorff dimension and showing it is usually strictly smaller than the Box dimension. For instance, the set determined by the pattern in Fig. 1 has Hausdorff dimension \( \log_2(1 + 2^{\log_2 3}) \); for a general pattern \( S \), the dimension is \( \log_m \left( \sum_{i=0}^{m-1} b_i \right) \), where \( b_i \) is the number of rectangles in row \( i \) of \( S \).

**Fig. 1.** Typical McMullen’s example

### 1.2. Sofic Affine-Invariant Sets

Let \( G = \langle V; E \rangle \) be a finite directed graph in which loops and multiple edges are allowed. Let \( D \) be the set of symbols, \( |D| = l \). Suppose the edges of \( G \) are labeled in symbols in \( D \) in a right resolving fashion: no two edge emanating from the same vertex marked the same symbol. Then the symbol sequences which arise from infinite paths in \( G \) form a sofic system on \( l \) symbols. That is

\[
\Omega = \{ \{ d_k \}_{k \geq 1} \mid \{ d_k \}_{k \geq 1} \text{ is an infinite path in } G, \text{ where } d_k \in D \}.
\]

**Definition 1.1.** Let \( G = \langle V; E \rangle \) be a directed graph and \( D = \{0,1, \cdots, n-1\} \times \{0,1, \cdots, m-1\} \) be the set of symbols. Suppose \( \Omega \subset \mathbb{D}^\mathbb{N} \) is the resulting sofic system. We call

\[
R_T(\Omega) = \left\{ \sum_{k=1}^{\infty} T^{-k} d_k \mid \{ d_k \}_{k \geq 1} \in \Omega \right\}
\]

a \( T \)-invariant sofic set.

In some papers such sets are called recurrent sets\([2, 3]\) or graph-directed sets\([12]\).

The edges in \( G \) are conveniently represented by the adjacency matrix \( A \) where for any two vertices \( v, w \) in \( G \), \( A(v, w) \) is the number of edges in \( E \) from \( v \) to \( w \). Now we construct \( m \) matrices \( A_0, A_1, \cdots, A_{m-1} \) as follows: \( A_j \) is a \( |V| \times |V| \) matrix such that for vertices \( v, v' \) in \( G \), \( A_j(v, v') \) denotes the number of edges from \( v \) to \( v' \) in \( G \) such that the second coordinate of their label is \( j \). R. Kenyon and Y. Peres\([10]\) studied the dimensions of the sofic affine-invariant sets and showed that

\[
\dim H R_T(\Omega) = \lim_{k \to \infty} \frac{1}{k} \log_m \sum_{0 \leq i_1, \cdots, i_k \leq m-1} \| A_{i_k} \cdot A_{i_{k-1}} \cdots A_{i_1} \|^\alpha,
\]

where \( \alpha = \frac{\log m}{\log n} \leq 1 \) (the choice of norm is clearly immaterial).

**Example 1.1.** Here we give an example of sofic invariant set\([10, Example 4.2]\). Let

\[
G = \langle V, E \rangle \text{ and } V = \{a, b\}
\]

and the label set

\[
D = \{0, 1, 2\} \times \{0, 1\}.
\]

The graph is shown by Fig. 2a.
A sofic system can also be illustrated by a group patterns. For this example the sofic system is illustrated by Fig. 2b.

The adjacency matrix of this sofic system is

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]

and

\[
A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

[10] has calculated that

\[
\dim_H K = 0.883127\ldots \quad \text{and} \quad \dim_B K = 0.8875138\ldots.
\]

1.3. The Main Result

Example 1.2. Kenyon and Peres[10] have considered the following one parameter family of sets. Let \(0 \leq u \leq 1 = 2\) and let \(K_u\) be the attractor for the affine maps

\[
f_j \begin{pmatrix} x \\ y \end{pmatrix} = T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + d_j, \quad d_j \in D,
\]

where \(T = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\) and \(D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ u \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}\). See Fig. 3a.

By a general result of Deliu et al[4] and Falconer[7], for all \(0 \leq u \leq 1/2\),

\[
\dim_B K_u = 2 - \log_3 2 = 1.36907\ldots
\]

For the Hausdorff dimension, they observed that \(2K_{1/2}\) is a sofic set determined by the substitution in Fig. 3b and clearly

\[
\dim_H K_{1/2} = \dim_H 2K_{1/2} = 1.36629\ldots.
\]
They claimed that the method by involving more work might apply to calculating $\dim_H K_u$ for any rational value of $u$.

In this paper we introduce a method which not only may apply to the above case, but also is possible to deal with the self-affine sets with overlaps.

Let $T = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ with $1 \leq m \leq n$, $D = \{d_1, d_2, \ldots, d_N\} \subset \mathbb{R}^2$. Let $d_j = (d_j(x), d_j(y))$.

Without loss of generality we can assume that $0 \leq d_j(x) \leq n-1$, $0 \leq d_j(y) \leq m-1$.

Let \( \{f_j\}_{j=1}^N \) be a family of contraction maps defined by

\[
 f_j\left(\frac{x}{y}\right) = T^{-1}\left(\frac{x}{y} + d_j\right), \quad d_j \in D, \tag{1.1}
\]

and $K(T, D)$ by the associated invariant set such that

\[
 K(T, D) = \bigcup_{j=1}^N f_j(K(T, D)).
\]

The typical pattern of these sets is shown in Fig. 4. Usually they are self-affine sets with overlaps. Generally, it is very difficult to calculate the dimensions of self-similar sets with overlaps, and it is awfully hard for self-affine sets with overlaps.

In this paper we also require that the digits set $D$ is rational, that is, the coordinates of digits of $D$ are rational (the case that $D$ is irrational, that is, some coordinates of digits of $D$ may be irrational, seems difficult). Our main result is

**Theorem 1.1.** Let $D = \{d_1, \ldots, d_N\}$ be rational and $\{f_j\}_{j=1}^N$ be a function system defined as in (1.1). Then there is a sofic affine invariant set $R(\Omega)$ such that $R(\Omega) = K(T, D)$.

In fact we develop an algorithm to construct a sofic system $\Omega$ (that is, the graph directed set $G$ with labeled edge) such that $K(T, D) = R(\Omega)$ and the Hausdorff dimension follows. [9] has used this idea to study the self-similar sets with overlaps, but indeed it also works for the setting of self-affine sets.

**Example 1.3.** Let $T = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and

\[
 D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/1 \\ 1 \end{pmatrix} \right\}.
\]

See Fig. 5a.
We will show that the equivalent sofic system is

![Fig. 5a](image)

The adjacency matrix is

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 0 & 1 \\
\end{pmatrix}
\]

and

\[
A_0 = \begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{pmatrix},
\]

\[
A_1 = \begin{pmatrix}
0 & 1 & 1 \\
0 & 2 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

The details are in Subsection 2.3.

**Example 1.4.** Let \( T = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \) and

\[
D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.
\]

See Fig. 6a.
We will show that the equivalent sofic system is

The adjacency matrix is

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 2 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

and

\[
A_0 = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
A_1 = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

§2. Algorithm to Construct Sofic Invariant Sets

2.1. Label of Cylinders

Let \( T = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix} \) and \( D = \{ d_1, \cdots, d_N \} \) be rational. \( \{ f_i \}_{i=1}^N \) is the function system defined in (1.1) and denote by \( K(T, D) \) its invariant sets. \( K(T, D) \) most likely is a self-affine set with overlaps. Even if \( K(T, D) \) has no overlaps, usually it is not McMullen’s affine carpets (see Fig. 4) and thus very hard to calculate its dimensions.

In this section we develop an algorithm to construct a sofic invariant set \( R(\Omega) \) such that \( K(T, D) = R(\Omega) \).

Let \( D_0 = \{ 0, 1, \cdots, n-1 \} \times \{ 0, 1, \cdots, m-1 \} \). Let \( \{ g_j \}_{j=1}^{nm} \) be a function system defined by

\[
g_j \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = T^{-1} \left( \begin{pmatrix} x \\ y \end{pmatrix} + e_j \right), \quad e_j \in D_0.
\]
Then the associated invariant set $K(T, D_0)$ is exactly the unit square $[0, 1] \times [0, 1]$, denoted by $K_0$. We call 

$$g_{i_1 \cdots i_k}(K_0) = g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_k}(K_0), \quad 1 \leq i_j \leq nm,$$

a $k$-cylinder of $K(T, D_0)$. Denote the collection of all $k$-cylinder of $K(T, D_0)$ by $C_{D_0,k}$. Similarly we call 

$$f_{i_1 \cdots i_k}(K_0) = f_{i_1} \circ f_{i_2} \cdots \circ f_{i_k}(K_0), \quad 1 \leq i_j \leq N,$$

a $k$-cylinder of $K(T, D)$. Denote the collection of all $k$-cylinders of $K(T, D)$ by $C_{D,k}$. By convention, $K_0$ is defined as a 0-cylinder. A $k$-cylinder of $K(T, D)$ or $K(T, D_0)$ is a rectangle of size $n^{-k} \times m^{-k}$ and sides parallel the coordinate axis.

Now we concentrate on the system \{g_i\}_{i=1}^N. We study how the $k$-cylinders of \{g_i\}_{i=1}^N intersect the $k$-cylinders of \{f_i\}_{i=1}^N.

**Definition 2.1.** Suppose $\tau \in C_{D_0,k}$. The neighborhood of $\tau$ with respect to $C_{D,k}$ is defined as 

$$N(\tau) = \{ \delta : \delta \in C_{D,k} \text{ and the interior of } \delta \cap \tau \text{ not empty} \}.$$ 

Notice that a $k$-cylinder of $C_{D,k}$, say $\delta$, is a rectangle of size $n^{-k} \times m^{-k}$ and its sides parallel to the coordinates axis, so it is completely determined by its left-lower corner, denoted this point by $\delta$. Now we give each cylinder of $C_{D,k}$ a label.

**Definition 2.2.** The label of $\tau \in C_{D_0,k}$ is defined as 

$$L(\tau) = \{ T^k(\delta - \bar{\tau}) : \delta \in N_k(\tau) \}.$$

Denote 

$$L = \bigcup_{k \geq 0} \{ L(\tau) : \tau \in C_{D_0,k} \}.$$

**Proposition 2.1.** $L$ is a finite set.

**Proof.** For $\tau \in C_{D_0,k}$, $\delta \in C_{D,k}$,

$$\bar{\tau} = \sum_{j=1}^{k} T^{-j}e_{i_j}, \quad e_{i_j} \in D_0;$$

$$\bar{\delta} = \sum_{j=1}^{k} T^{-j}d_{i_j}, \quad d_{i_j} \in D.$$ 

So 

$$T^k(\bar{\delta} - \bar{\tau}) = \sum_{j=1}^{k} T^{k-j}(d_{i_j} - e_{i_j}).$$

Suppose that $M$ is an integer such that $M d_{i_j}$ is integer for each $d_{i_j} \in D$. Then the coordinates of $MT^k(\bar{\delta} - \bar{\tau})$ are integers. $\tau \cap \delta \neq \emptyset$ implies 

$$T^k(\delta - \bar{\tau}) \in [-n^{-1}, n^{-1}] \times [-m^{-1}, m^{-1}],$$

so 

$$MT^k(\delta - \bar{\tau}) \in [-M(n^{-1}, n^{-1})] \times [-M(m^{-1}), m^{-1}].$$

Then for any $\tau \in C_{D_0,k}$,

$$L(\tau) \subseteq \frac{([-M(n^{-1}, n^{-1})] \times [-M(m^{-1}), m^{-1}]) \cap \mathbb{Z}^2}{M}.$$ 

Notice the right side of the above formula is a finite set, so $L$ is also a finite set.
2.2. Sofic Invariant Set

It is possible that for some cylinder \( \tau \), it does not meet any cylinder of \( K(T,D) \) and it has no contribution. This means that \( \emptyset \) is an element of \( L \). Delete empty from \( L \) we get \( L^* \), that is, \( L^* = L \setminus \{ \emptyset \} \).

A \( k \)-cylinder of \( K(T,D_0) \) will be partitioned into \( n \times m \) \((k+1)\)-cylinders. Marking each sub-cylinder by its label and deleting the sub-cylinders with label \( \emptyset \), we get a pattern.

**Proposition 2.2.** Suppose that \( \tau \) and \( \tau' \) are two cylinders of \( K(T,D_0) \). If \( L(\tau) = L(\tau') \), then the pattern of \( \tau \) coincides with the pattern of \( \tau' \).

**Proof.** Suppose that \( \tau \) is a \( k \)-cylinder and its label is \( L(\tau) \). \( \tau \) can be partitioned into \( n \times m \) \((k+1)\)-cylinders. The neighbourhood of \( \tau \) is

\[
N(\tau) = \{ T^{-k}e + \bar{\tau} : e \in L(\tau) \}.
\]

For a vector \( v \) of the plane, we denote \( v_x \) the \( x \)-coordinate of \( v \) and \( v_y \) the \( y \)-coordinate of \( v \). Let \( f = (f_x,f_y) \) be one of the sub-cylinders, \( 0 \leq f_x \leq m - 1 \), \( 0 \leq f_y \leq n - 1 \). Denote

\[
N = \{ e + T^{-(k+1)}d : e \in N(\tau), \ d \in D \}.
\]

Then the neighbour of \( f \) can only come from \( N \), thus the label of \( f \) is

\[
L(f) = \{ T^{k+1}(e + \bar{\tau} + T^{-(k+1)}f) : e \in C_{D_0,k+1}, \ e \cap f \neq \emptyset \}.
\]

Therefore we get

\[
L(f) = \{ Te + d : e \in L(\tau), \ d \in D, \ \|Te + d\|_x \leq 1, \ |Te + d|_y \leq 1 \}.
\]

The labels of the sub-cylinders are completely determined by \( L(\tau) \), so the pattern of \( \tau \) is completely determined by \( L(\tau) \).

Suppose that \( L^* \) has \( l \) elements. By the above proposition we will obtain \( l \) patterns. These patterns determine a sofic system \( \Omega \) and a sofic invariant set \( R(\Omega) \). Our last step is to show that \( K(T,D) \) and \( R(\Omega) \) coincide.

**Theorem 2.1.** With the above notation we have

\[
K(T,D) = R(\Omega).
\]

**Proof.** We abbreviate \( K(T,D) \) as \( K \) and \( R(\Omega) \) as \( R \). Let \( K(k) \) be the union of the \( k \)-cylinders of \( K \) and \( R(k) \) be the \( k \)-th iteration of the sofic system. Then by [5],

\[
K(k) \to K \quad \text{and} \quad R(k) \to R \quad \text{in} \quad d_H,
\]

where \( d_H \) denotes the Hausdorff metric. By the construction of \( R(\Omega) \), we have

\[
d_H (K(k), R(k)) \leq m^{-k}.
\]

So for any \( k \in \mathbb{N} \),

\[
d_H (K, R) \leq d_H (K, K(k)) + d_H (R, R(k)) + d_H (K(k), R(k)) \leq 3m^{-k},
\]

thus \( d_H (K, R) = 0 \) which yields \( K = R \).

2.3. Algorithm
Now we make the algorithm more practical.

(1) We begin with the 0-cylinder whose label is \( \{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \} \). We put it as the initial vertex into \( V \).

(2) We work out the patterns of all vertices in \( V \) by the formula (2.1). We may get some new types of label. Regard them as new vertices and add them to \( V \).

(3) Repeat the procedure 2 until there is no new type of label occurring. By Proposition 2.1 the procedure will stop at finite steps.

Now we get the vertex set \( V \) and the patterns, and thus get the desired sofic system.

**Example 2.1.** Using the above algorithm we calculate that \( V = \{ A, B, C \} \), where

\[
A = \left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right\}, \\
B = \left\{ \left( \begin{array}{c} 0 \\ -\frac{1}{2} \end{array} \right) \right\}, \\
C = \left\{ \left( \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right) \right\}.
\]

The corresponding patterns are exactly the same as Fig. 3b.

**Example 2.2.** Using the above algorithm we calculate that \( V = \{ A, B, C \} \), where

\[
A = \left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right\}, \\
B = \left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} \frac{2}{3} \\ 0 \end{array} \right) \right\}, \\
C = \left\{ \left( -\frac{1}{3} \right) \right\}.
\]

The corresponding patterns are depicted by Fig. 5b.

**Example 2.3.** We calculate that \( V = \{ A, B, C, D, E \} \), where

\[
A = \left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right\}, \\
B = \left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} \frac{2}{3} \\
\frac{1}{3} \end{array} \right) \right\}, \\
C = \left\{ \left( -\frac{1}{3} \right) \right\}, \\
D = \left\{ \left( -\frac{2}{3} \right) \right\}, \\
E = \left\{ \left( -\frac{1}{3} \right) \right\}.
\]

The corresponding patterns are depicted by Fig. 6b.

**References**


