Growth and distortion theorems for a subclass of holomorphic mappings *

Qinghua Xu[1 †] , Taishun Liu[2] and Huiming Xu[3]

E–mail address: xuqh@mail.ustc.edu.cn, lts@ustc.edu.cn.

(1.College of Mathematics and Information Science, JiangXi Normal University, NanChang 330022, China ;

2.Department of Mathematics, Huzhou Teacher's College, Huzhou 313000, China);

3.College of Mathematics, Physics and Information Engineering, ZheJiang Normal University, JinHua 321004, China)

Abstract Let X be a complex Banach space with norm $\|\cdot\|$, B be the unit ball in X. In this paper, we introduce a class of holomorphic mappings $\mathcal{M}_g$ on B. Let $f(x)$ be a normalized locally biholomorphic mappings on B such that $(Df(x))^{-1}f(x) \in \mathcal{M}_g$ and $x = 0$ is the zero of order $k+1$ of $f(x) - x$. We investigate the growth theorem for $f(x)$. As applications, the distortion theorems for the Jacobian matrix $J_f(z)$ are obtained, where $f(z)$ belongs to the subclasses of starlike mappings defined on the unit polydisc D^n in $\mathbb{C}^n$. These results unify and generalize many known results.

Keywords: Growth theorem, Distortion theorem, Subclasses of starlike mappings

Mathematics Subject Classification (2000): Primary 32H02; Secondary 30C45

1. Introduction

In the case of one complex variable, the following growth, distortion theorem is well-known [1].

Theorem A. *Let f be a normalized univalent holomorphic function on the unit disc D in $\mathbb{C}$. Then*

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in D, \tag{1}$$

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in D.$$

*This work was supported by NNSF of China (Grant Nos. 11261022, 11061015, 11031008), the Jiangxi Provincial Natural Science Foundation of China (Grant No. 2010GZS0096), Natural Science Foundation of Department of Education of Jiangxi Province, China (Grant No. GJJ 12177) and the Zhejiang Provincial Natural Science Foundation of China (Grant No. Y6110053).

†Corresponding author

However, in the case of several complex variables, Cartan [2] pointed out that the above theorem does not hold.

Barnard et al.[3] and Chuaqui [4] extended the growth theorem (1) to normalized starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$. After that, many mathematicians investigate the growth theorems for subclasses of starlike mappings ([5], [6], [7], [8]).

As for the distortion theorems for subclasses of normalized biholomorphic mappings, Pfaltzgraff and Suffridge [9] obtained a distortion result for a subclass of starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$. Xu and Liu ([10], [11]) obtained a sharp distortion theorem for a subclass of biholomorphic mappings. Recently, Liu et al.([12], [13], [14]) obtained a distortion theorem for quasi-convex mappings, starlike mappings and a subclass of quasi-convex mappings on the unit polydisc D^n in $\mathbb{C}^n$, respectively.

In this paper, we shall obtain growth theorem for a class of biholomorphic mappings. From it, the distortion theorems for subclasses of starlike mappings are obtained. These results generalize the related works of several authors.

Let X be a complex Banach space with norm $\|\cdot\|$, X^* be the dual space of X, B be the unit ball in X, D be the open unit disc in $\mathbb{C}$, D^n represent the open unit polydisk in $\mathbb{C}^n$. Let $(\partial D)^n(0,r)$ be the distinguished boundary of the polydisc of radius r with the center 0. For each $x \in X \setminus \{0\}$, we define $T(x) = \{l_x \in X^* : \|l_x\| \leq 1, l_x(x) = \|x\|\}$. According to the Hahn-Banach theorem, $T(x)$ is nonempty. Let $H(B)$ be the set of all holomorphic mappings from B into X. Notice that for fixed $x \in X$, $\forall \alpha (\neq 0) \in \mathbb{C}$, when l_x is chosen and fixed, then $\|\frac{|\alpha|}{\alpha} l_x\| = \|l_x\| \leq 1$, and $\frac{|\alpha|}{\alpha} l_x(\alpha x) = \frac{|\alpha|}{\alpha} \alpha l_x(x) = |\alpha|\|x\| = \|\alpha x\|$, so we can set $l_{\alpha x} = \frac{|\alpha|}{\alpha} l_x$. A holomorphic mapping $f : B \to X$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$(When $X = \mathbb{C}^n$, $Df(x)$ is denoted by $J_f(z)$, where $J_f(z)$ is the Jacobian matrix of f at point z) has a bounded inverse for each $x \in B$. We say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X. Let $S(B)$ be the set of all normalized biholomorphic mappings. We say that f is starlike if f is biholomorphic on B and $f(B)$ is starlike with respect to the origin. Let $S^*(B)$ be the set of normalized starlike mappings on B.

Definition 1. Let $g \in H(D)$ be a biholomorphic function such that $g(0) = 1$, $g(\bar{\xi}) = \overline{g(\xi)}$, for $\xi \in D$, $\Re e g(\xi) > 0$ on $\xi \in D$, and assume g satisfies the following conditions for $r \in (0,1)$:

$$\begin{cases} \min\limits_{|\xi|=r} |g(\xi)| = \min\limits_{|\xi|=r} \Re e g(\xi) = g(-r) \\ \max\limits_{|\xi|=r} |g(\xi)| = \max\limits_{|\xi|=r} \Re e g(\xi) = g(r). \end{cases} \tag{2}$$

We define $\mathcal{M}_g$ to be the class of mappings given by

$$\mathcal{M}_g = \left\{ p \in H(B) : p(0) = 0, Dp(0) = I, \frac{\|x\|}{l_x(p(x))} \in g(D), x \in B \backslash \{0\}, l_x \in T(x) \right\}.$$

In a slightly different manner, Definition 1 was considered by Kohr [15] on B^n and by Graham et al.[16] on the unit ball with respect to an arbitrary norm on $\mathbb{C}^n$. The set $\mathcal{M}_g$ has been important in the study of certain problems related to Löwner chains on the unit ball in $\mathbb{C}^n$ (see [17]).

Let $S_g^*(B)$ denote the subset of $S^*(B)$ consisting of those normalized locally biholomorphic mappings f such that $[Df(x)]^{-1}f(x) \in \mathcal{M}_g$.

Definition 2. Let $0 \leq \alpha < 1$. A normalized locally biholomorphic mapping $f \in H(B)$ is said to be starlike of order α if

$$[Df(x)]^{-1}f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1+(1-2\alpha)\zeta}{1-\zeta}, \ \zeta \in D$.

We denote by $S_\alpha^*(B)$ the set of all starlike mappings of order α on B.

The following definition due to Roper and Suffridge [18].

Definition 3. Suppose $f : B \to X$ is a normalized locally biholomorphic mapping, denote

$$G_f(\alpha, \ \beta) = \frac{2\alpha}{l_u[(Df(\alpha u)^{-1}(f(\alpha u)) - f(\beta u))]} - \frac{\alpha+\beta}{\alpha-\beta}.$$

If

$$\Re e G_f(\alpha, \ \beta) \geq 0, \ \forall u \in \partial B, \ \alpha, \ \beta \in D,$$

then f is said to be a quasi-convex mapping of type A on B.

Let $Q_A(B)$ denote the class of quasi-convex mappings of type A on B.

Definition 4. (see [19].) A normalized locally biholomorphic mapping $f : B \longrightarrow X$ is said to be quasi-convex if

$$\Re e l_x[(Df(x))^{-1}(f(x) - f(\xi x))] \geq 0, \quad \forall x \in B, \quad \xi \in \overline{D}, \ \ l_x \in T(x).$$

Let $Q(B)$ denote the class of quasi-convex mappings on B.

Remark 1. In [19], it is proved that $Q(B) = Q_A(B)$. Roper and Suffridge [18] also proved that $Q_A(B) \subset S^*_{\frac{1}{2}}(B)$, and therefore f satisfies the following relation

$$\left| \frac{l_x[(Df(x))^{-1}f(x)]}{\|x\|} - 1 \right| \leq 1. \tag{3}$$

From (3), we obtain

$$[Df(x)]^{-1}f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1}{1-\zeta}, \ \zeta \in D$. Hence, we have

$$Q(B) = Q_A(B) \subset S_g^*(B) \ \text{ with } \ \ g(\zeta) = \tfrac{1}{1-\zeta}, \ \ \zeta \in D.$$

In 2006, Liu and Xu [20] defined quasi-convex mappings of order α, which is a proper subset of the quasi-convex mappings.

Definition 5. Suppose $\alpha \in [0,1)$. A normalized locally biholomorphic mapping $f : B \longrightarrow X$ is said to be a quasi-convex mapping of order α if

$$\Re e l_x[(Df(x))^{-1}(f(x) - f(\xi x))] \geq \alpha(1 - \Re e\xi)\|x\|, \quad \forall x \in B, \quad \xi \in \overline{D}, \ \ l_x \in T(x).$$

Let $Q_\alpha(B)$ denote the class of quasi-convex mappings on B.

The following proposition is a well-known result in one complex variable.

Proposition 1. If $g : D \longrightarrow D$ is a holomorphic function, $z = 1$ is not a singular point of g and $g(0) = 0, \ g(1) = 1$. Then $g'(1) \geq 1$.

Remark 2. Let $x \in B \setminus \{0\}$, and denote that

$$h_x(\xi) = l_x[(Df(x))^{-1}(f(x) - f(\xi x))] - \alpha(1 - \xi)\|x\|, \quad \xi \in \overline{D}.$$

By Definition 5, we know that as a function of ξ, $\Re e h_x(\xi) \geq 0$ and harmonic in D, so by the minimal value principle we have $\Re e h_x(0) > 0$. From this we obtain

$$| \ h_x(0) - h_x(\xi) \ | \leq | \ \overline{h_x(0)} + h_x(\xi) \ | \ .$$

Let

$$g(\xi) = \frac{\overline{h_x(0)}}{h_x(0)} \left(\frac{h_x(0) - h_x(\xi)}{\overline{h_x(0)} + h_x(\xi)} \right)$$

for $\xi \in \overline{D}$, then $g : D \longrightarrow D$ is a holomorphic function of ξ with $g(0) = 0$, and $\xi = 1$ is not a singularity of $g(\xi)$, $g(1) = 1$. According to Proposition 1, we have

$$1 \leq g'(1) = \frac{\overline{h_x(0)}}{h_x(_0)} \cdot \frac{(1-\alpha)\|x\|\overline{h_x(0)} + (1-\alpha)\|x\|h_x(0)}{\overline{h_x(0)^2}} = \frac{2(1-\alpha)\|x\|\Re e h_x(0)}{| \ h_x(0) \ |^2}.$$

That is ,

$$| \ h_x(0) - (1-\alpha)\|x\| \ | \leq (1-\alpha)\|x\|.$$

More explicitly, it is

$$| \ l_x(Df(x))^{-1}f(x) - \|x\| \ | \leq (1-\alpha)\|x\|.$$

That is the same as

$$\left| \frac{l_x[(Df(x))^{-1}f(x)]}{(1-\alpha)\|x\|} - \frac{1}{1-\alpha} \right| \leq 1. \tag{4}$$

From (4), it follows that

$$[Df(x)]^{-1}f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1}{1-(1-\alpha)\zeta}$, $\zeta \in D$. Hence, we obtain that

$$Q_\alpha(B) \subset S_g^*(B) \ \text{ with } \ g(\zeta) = \tfrac{1}{1-(1-\alpha)\zeta}, \ \ \zeta \in D.$$

Let $f \in H(B)$ and let k be a positive integer. We say that $z = 0$ is a zero of order k of $f(z)$ if $f(0) = 0, \cdots, D^{k-1}f(0) = 0$ and $D^k f(0) \neq 0$.

Also, we denote by $S^*_{k+1}(B)$(respectively $S^*_{g,\ k+1}(B)$, $S^*_{\alpha,\ k+1}(B)$, $Q_{k+1}(B)$, $Q_{\alpha,\ k+1}(B)$), the subset of $S^*(B)$(respectively $S^*_g(B)$, $S^*_\alpha(B)$, $Q(B)$, $Q_\alpha(B)$) of mappings f such that $x=0$ is a zero of order $k+1$ of $f(x)-x$.

2. Growth theorem

Lemma 1. (See [21].) *Suppose $x(t):[0,1]\to X$ is differentiable at the point s which belongs to $[0,1]$, and $\|x(t)\|$ is differentiable at the point s with respect to t. Then*

$$\Re e\left[l_{x(t)}\left(\frac{dx(t)}{dt}\right)\right]\Big|_{t=s}=\frac{d(\|x(t)\|)}{dt}\Big|_{t=s}.$$

Lemma 2. (See [22].) *Suppose f is a starlike mapping on B, $x\in B\backslash\{0\}$, $x(t)=f^{-1}(tf(x))(0\le t\le 1)$. Then*
(a) $\|x(t)\|$ is strictly increasing on $[0,1]$ with respect to t;
(b) $\|f(x)\|=\lim\limits_{t\to 0}\frac{\|x(t)\|}{t}$, $\frac{dx(t)}{dt}=\frac{1}{t}[Df(x(t))]^{-1}f(x(t))\,,t\in(0,1)\,.$
Lemma 3. (See [23].) *If $f\in H(D)$, g is a biholomorphic function on D, $f(0)=g(0), f'(0)=\cdots=f^{(k-1)}(0)=0$, and $f\prec g$. Then*

$$f(rD)\subseteq g(r^kD),\quad r\in(0,1),\ rD=\{\xi\in\mathbb{C}:|\xi|<r\}.$$

Using Lemma 3, we can prove the following.
Lemma 4. *Let $g:D\to\mathbb{C}$ satisfy the conditions of Definition 1. If $h\in\mathcal{M}_g$ and $x=0$ is the zero of order $k+1$ of $h(x)-x$, then*

$$\frac{\|x\|}{g(\|x\|^k)}\le\Re e l_x((h(x))\le|l_x(h(x))|\le\frac{\|x\|}{g(-\|x\|^k)}\tag{5}$$

for all $x\in B$.

Proof. Fix $x\in B\backslash\{0\}$, and denote $x_0=\frac{x}{\|x\|}$. Let $p:D\longrightarrow\mathbb{C}$ be given by

$$p(\xi)=\begin{cases}\frac{\xi}{l_x(h(\xi x_0))}, & \xi\neq 0,\\ 1, & \xi=0.\end{cases}$$

Then $p\in H(D)$, $p(0)=g(0)=1$, and since $h\in\mathcal{M}_g$, we deduce that

$$p(\xi)=\frac{\xi}{l_x(h(\xi x_0))}=\frac{\xi}{l_{x_0}(h(\xi x_0))}=\frac{\|\xi x_0\|}{l_{\xi x_0}(h(\xi x_0))}\in g(D),\quad \xi\in D.$$

Let $\psi(\xi)=\frac{1}{p(\xi)}$. This implies that $\psi(\xi)\in\frac{1}{g}(D)$ for all $\xi\in D$. Since $\psi(0)=\frac{1}{g}(0)=1$, we have $\psi\prec\frac{1}{g}$.

According to hypothesis of Lemma 4, we deduce that

$$\psi(\xi)=1+\sum_{m=k+1}^{\infty}\frac{l_z(D^mh(0)(z_0^m))}{m!}\xi^{m-1}.$$

It is easy to see that the function $\psi(\xi)$ satisfies the conditions of Lemma 3, hence we obtain

$$\psi(rD) \subseteq \frac{1}{g}(r^k D), \quad r \in (0,1),\ rD = \{\xi \in \mathbb{C} : |\xi| < r\}.$$

On the other hand, combining the maximum and minimum principles for harmonic functions with (2), we deduce that

$$\frac{1}{g(|\xi|^k)} \leq \Re e\psi(\xi) \leq |\psi(\xi)| \leq \frac{1}{g(-|\xi|^k)}, \quad \xi \in D.$$

Setting $\xi = \|x\|$ in the above relation, we obtain (5), as desired. This completes the proof. $\square$

In [6], Hamada and Honda have recently obtained a sharp growth result for mappings in the family $S^*_{g,\ k+1}(B)$, and g satisfies a slightly different assumption than that in Definition 1. Stimulated by [6], we are now able to obtain the following growth result for the set $f \in S^*_{g,\ k+1}(B)$. This result generalizes [16, Theorem 2.2], [15, Theorem 2.3] and [10, Theorem 2].

Theorem 1. *Let $g : D \to \mathbb{C}$ satisfy the conditions of Definition 1. If $f \in S^*_{g,\ k+1}(B)$, then*

$$\|x\| \exp \int_0^{\|x\|} \Big[g(-y^k) - 1\Big] \frac{dy}{y} \leq \|f(x)\| \leq \|x\| \exp \int_0^{\|x\|} \Big[g(y^k) - 1\Big] \frac{dy}{y}, \quad x \in B. \tag{6}$$

Proof. Since $f \in S^*_{g,\ k+1}(B)$, we deduce from Lemma 4 that

$$\frac{\|x\|}{g(\|x\|^k)} \leq \Re e l_x(Df(x))^{-1} f(x) \leq \frac{\|x\|}{g(-\|x\|^k)} \tag{7}$$

for all $x \in B$. Fix $x \in B \setminus \{0\}$, let $x(t) = f^{-1}(tf(x))(0 \leq t \leq 1)$. According to (a) of Lemma 2, we obtain that $\|x(t)\|$ is strictly increasing on [0,1]. Hence, $\|x(t)\|$ is differentiable on [0,1] a.e. . From Lemmas 1, 2(b) and (7), we deduce that for $t \in (0,1]$

$$\frac{\|x(t)\|}{g(\|x(t)\|^k)} \leq t\frac{d\|x(t)\|}{dt} \leq \frac{\|x(t)\|}{g(-\|x(t)\|^k)}, \tag{8}$$

and we may rewrite (8) as

$$\frac{g(-\|x(t)\|^k)}{\|x(t)\|}\frac{d\|x(t)\|}{dt} \leq \frac{1}{t} \leq \frac{g(\|x(t)\|^k)}{\|x(t)\|}\frac{d\|x(t)\|}{dt}.$$

Integrating both sides of the above inequalities with respect to t and making a change of variable, we obtain

$$\int_{\|x(\varepsilon)\|}^{\|x\|} \frac{g(-y^k)\}dy}{y} = \int_\varepsilon^1 \frac{g(-\|x(t)\|^k))\}}{\|x(t)\|}\frac{d\|x(t)\|}{dt}dt \leq \int_\varepsilon^1 \frac{1}{t}dt,$$

and

$$\int_{\|x(\varepsilon)\|}^{\|x\|} \frac{g(y^k)dy}{y} = \int_\varepsilon^1 \frac{g(\|x(t)\|^k}{\|x(t)\|}\frac{d\|x(t)\|}{dt}dt \geq \int_\varepsilon^1 \frac{1}{t}dt,$$

where $0 < \varepsilon < 1$. It is elementary to verify that

$$\log \frac{\|x(\varepsilon)\|}{\varepsilon} \geq \int_{\|x(\varepsilon)\|}^{\|x\|} \Big[g(-y^k) - 1\Big] \frac{dy}{y} + \log \|x\|, \tag{9}$$

and

$$\log \frac{\|x(\varepsilon)\|}{\varepsilon} \leq \int_{\|x(\varepsilon)\|}^{\|x\|} \Big[g(y^k) - 1\Big] \frac{dy}{y} + \log \|x\|. \tag{10}$$

If we now let $\varepsilon \longrightarrow 0+$ in the above inequalities (9), (10) and use Lemma 2(b), we have

$$\|x\| \exp \int_0^{\|x\|} \Big[g(-y^k) - 1\Big] \frac{dy}{y} \leq \|f(x)\| \leq \|x\| \exp \int_0^{\|x\|} \Big[g(y^k) - 1\Big] \frac{dy}{y}, \quad x \in B,$$

as claimed. This completes the proof of Theorem 1. $\square$

3. Distortion theorem

In this section, we will give distortion theorems for subclasses of starlike mappings along a unit direction in $S^*_{g\ k+1}(D^n)$.

Theorem 2. *Let $g : D \to \mathbb{C}$ satisfy the conditions of Definition 1. If $f \in S^*_{g,\ k+1}(D^n)$, then for any $z \in D^n \setminus \{0\}$, there exists a unit vector $\xi(z)$ $\left(\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, such that*

$$g(-\|z\|^k) \exp \int_0^{\|x\|} \Big[g(-y^k) - 1\Big] \frac{dy}{y} \leq \|J_f(z)\xi(z)\| \leq g(\|z\|^k) \exp \int_0^{\|x\|} \Big[g(y^k) - 1\Big] \frac{dy}{y}.$$

Proof. For $z \in D^n \setminus \{0\}$ and any $\xi \in (\partial D)^n(0, \|z\|)$, we have

$$|\xi_1| = |\xi_2| = \cdots = |\xi_n| = \|z\|.$$

Note that

$$l_\xi = (0, \cdots, 0, \frac{\|\xi\|}{\xi_i}, 0, \cdots, 0).$$

Set $w(z) = J_f^{-1}(z)f(z)$. Then there exists an i such that

$$\begin{aligned}
\|w(z)\| = |w_i(z)| &\leq \max_{\xi \in (\partial D)^n(0,\|z\|)} |w_i(\xi)| \\
&= \max_{\xi \in (\partial D)^n(0,\|z\|)} \left|\frac{\|\xi\|}{\xi_i} w_i(\xi)\right| \\
&= \max_{\xi \in (\partial D)^n(0,\|z\|)} |l_\xi[w(\xi)]| \\
&= \max_{\xi \in (\partial D)^n(0,\|z\|)} |l_\xi[J_f^{-1}(\xi)f(\xi)]|.
\end{aligned}$$

According to Lemma 4, we have

$$|l_\xi(J_f^{-1}(\xi)f(\xi)| \leq \frac{\|\xi\|}{g(-\|\xi\|^k)} = \frac{\|z\|}{g(-\|z\|^k)},$$

and thus

$$\|J_f^{-1}(z)f(z)\| = \|w(z)\| \leq \frac{\|z\|}{g(-\|z\|^k)}. \tag{11}$$

On the other hand, $\|l_z\| \leq 1$ and Lemma 4 show that

$$\|J_f^{-1}(z)f(z)\| \geq \|l_z[J_f^{-1}(z)f(z)]\| \geq \frac{\|z\|}{g(\|z\|^k)}. \tag{12}$$

By combining (11) and (12), we have

$$\frac{\|z\|}{g(\|z\|^k)} \leq \|J_f^{-1}(z)f(z)\| \leq \frac{\|z\|}{g(-\|z\|^k)}. \tag{13}$$

Set $\xi(z)$ $\left(\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, where $z \in D^n \setminus \{0\}$. In view of Theorem 1, we have

$$\|z\| \exp \int_0^{\|z\|} \Big[g(-y^k) - 1\Big] \frac{dy}{y} \leq \|f(z)\| \leq \|z\| \exp \int_0^{\|z\|} \Big[g(y^k) - 1\Big] \frac{dy}{y}, \quad z \in D^n. \tag{14}$$

Consequently, from (13) and (14) we obtain

$$\begin{aligned} g(-\|z\|^k) \exp \int_0^{\|z\|} \Big[g(-y^k) - 1\Big] \frac{dy}{y} \leq & \|J_f(z)\xi(z)\| = \frac{\|f(z)\|}{\|J_f^{-1}(z)f(z)\|} \\ \leq & g(\|z\|^k) \exp \int_0^{\|z\|} \Big[g(y^k) - 1\Big] \frac{dy}{y}. \end{aligned}$$

as claimed. This completes the proof of Theorem 2. □

Now, we obtain the following corollaries from Theorem 2.

For $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in D$, we have the following distortion theorem along a unit direction of the polydisc D^n for the set $S_{k+1}^*(D^n)$. When $k = 1$, Corollary 1 was obtained by Liu et al. [13].

Corollary 1. *If $f \in S_{k+1}^*(D^n)$, then for any $z \in D^n \backslash \{0\}$, there exists a unit vector $\xi(z)$ $\left(\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, such that*

$$\frac{1 - \|z\|^k}{(1 + \|z\|^k)^{\frac{2}{k}+1}} \leq \|J_f(z)\xi(z)\| \leq \frac{1 + \|z\|^k}{(1 - \|z\|^k)^{\frac{2}{k}+1}}.$$

According to Remark 1, for $g(\zeta) = \frac{1}{1-\zeta}$, $\zeta \in D$, we have the following distortion theorem along a unit direction of the polydisc D^n for the set $Q_{k+1}(D^n)$. When $k = 1$, Corollary 2 was obtained by Liu et al. [12].

Corollary 2. *If $f \in Q_{k+1}(D^n)$, then for any $z \in D^n \backslash \{0\}$, there exists a unit vector $\xi(z)$ $\left(\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, such that*

$$\frac{1}{(1 + \|z\|^k)^{\frac{1}{k}+1}} \leq \|J_f(z)\xi(z)\| \leq \frac{1}{(1 - \|z\|^k)^{\frac{1}{k}+1}}.$$

According to Remark 2, for $g(\zeta)=\frac{1}{1-(1-\alpha)\zeta}$, $\zeta\in D$, we have the following distortion theorem along a unit direction of the polydisc D^n for the set $Q_{\alpha,\ k+1}(D^n)$ due to Wang et al. [14].

Corollary 3. *If $f\in Q_{\alpha,\ k+1}(D^n)$, then for any $z\in D^n\setminus\{0\}$, there exists a unit vector $\xi(z)$ $\left(\xi(z)=\frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, such that*

$$\frac{1}{(1+(1-\alpha)\|z\|^k)^{\frac{1}{k}+1}}\leq\|J_f(z)\xi(z)\|\leq\frac{1}{(1-(1-\alpha)\|z\|^k)^{\frac{1}{k}+1}}.$$

For $g(\zeta)=\frac{1+(1-2\alpha)\zeta}{1-\zeta}$, $\zeta\in D$, we have the following distortion theorem along a unit direction of the polydisc D^n for the set $S^*_{\alpha,\ k+1}(D^n)$. When $k=1$, Corollary 4 was obtained by Lu et al. [24].

Corollary 4. *If $f\in S^*_{\alpha,\ k+1}(D^n)$, then for any $z\in D^n\setminus\{0\}$, there exists a unit vector $\xi(z)$ $\left(\xi(z)=\frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, such that*

$$\frac{1-(1-2\alpha)\|z\|^k}{(1+\|z\|^k)^{\frac{2(1-\alpha)}{k}+1}}\leq\|J_f(z)\xi(z)\|\leq\frac{1+(1-2\alpha)\|z\|^k}{(1-\|z\|^k)^{\frac{2(1-\alpha)}{k}+1}}.$$

References

[1] Pommerenke, C., Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975

[2] Cartan, H., Sur la possibilité d'étendre aux fonctions de plusieurs variables complex, La théorie des fonctions univalents, Lecons sur les Fonctions Univalents ou Multivalentes, Paul Montel (ed.), GauthierVillars, Pairs, 1933

[3] Barnard, R. W., FitzGerald, C. H. and Gong, S., The growth and (1/4) -theorems for starlike mappings in $\mathbb{C}^n$, *Pacific J. Math.,* **150**, 13-22 (1991)

[4] Chuaqui, M., Applications of subordination chains to starlike mappings in $\mathbb{C}^n$, *Pacif. J. Math.,* **168**, 33-48 (1995)

[5] H. Hamada, H., Honda, T. and Kohr, G., Growth theorems and coefficient bounds for univalent holomorphic mappings which have parametric representation, *J. Math. Anal. Appl.,* **317**, 302-319 (2006)

[6] Hamada, H. and Honda, T., Sharp growth theorems and coefficient bounds for starlike mappings in several complex variables, *Chin. Ann. Math. Ser. B,* **29** (4), 353-368 (2008)

[7] Kohr, G., On starlikeness and strongly starlikeness of order α in $\mathbb{C}^n$, *Mathematica (Cluj).*, **40**(63), 95-109 (1998)

[8] Liu, T. S. and Liu, X. S., On the precise growth, covering and distortion theorem for normalized biholomorphic mappings, *J. Math. Anal. Appl.,* **295**, 404-417 (2004)

[9] Pfaltzgraff, J. A. and Suffridge, T. J., An extension theorem and linear invariant families generated by starlike maps, *Ann. Univ. Mariae Curie Sklodowska, Sect. A.,* **53**, 193-207 (1999)

[10] Xu, Q. H. and Liu, T. S., Biholomorphic mappings on bounded starlike circular domains, *J.Math.Anal. Appl.,* **366**, 153-163 (2010)

[11] Xu, Q. H. and Liu, T. S., Sharp growth, distortion theorems for a subclass of biholomorphic mappings, *Computers and Mathematics with Applications,* **59**, 3778-3784 (2010)

[12] Liu, T. S., Lu, J. and Wang, J. F., Distortion theorems of quasi-convex mappings in sveral complex variables, *Chin. Ann. Math. Ser.A,* **32**, 607-612 (2011)

[13] Liu, T. S., Wang, J. F. and Lu, J., Distortion theorems of starlike mappings along unit direction in C^n, *Taiwanese Journal of Mathemathics,* **15**, 2601-2608 (2011)

[14] Wang, J. F., Liu, T. S. and Lu, J., Growth and distortion theorems on subclasses of quasi-convex mappings in sveral complex variables, *Front. Math. China,* **6**, 931-944 (2011)

[15] Kohr, G., Using the method of Löwner chains to introduce some subclasses of biholomorphic mappings in $\mathbb{C}^n$, *Rev. Roum. Math. Pures Appl.,* **46**, 743-760 (2001)

[16] Graham, I., Hamada, H. and kohr, G., Parametric representation of univalent mappings in several complex variables, *Canad. J. Math.,* **54**(2), 324-351 (2002)

[17] Graham, I. and Kohr, G., Geometric Function Theory in One and Higher Dimensions, Marcel Dekker, New York, 2003

[18] Roper, K. and Suffridge, T. J., Convexity properties of holomorphic mappings in, *Trans.Amer.Math.Soc.,* **351**, 1803-1833 (1999)

[19] Zhang, W. J. and Liu, T. S., The growth and covering theorems for quasi-convex mappings on the unit ball in complex Banach spaces, *Sci. China Ser. A,* **45**, 1538-1547 (2002)

[20] Liu, T. S. and Xu, Q. H., Quasi-convex mappings of order α in the unit ball of a complex Banach space, *Sci. China Ser. A,* **49**, 1451-1457 (2006)

[21] Kato, T., Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan,* **19**, 508-520 (1967)

[22] Zhang, W. J. and Dong, D. Z., The growth and (1/4)-theorems for starlike mappings in a Banach space, *Chinese Ann. Math. Ser. A,* **13**, 417-423 (1992)

[23] Xu, Q. H. and Liu, T. S., The study for some subclasses of biholomorphic mappings by an unified method (in Chinese), *Chin. Quart. J. of Math.,* **2**, 9-19 (2006)

[24] Lu, J., Liu, T. S. and Wang, J. F., Distortion theorems for subclasses of starlike mappings along a unit direction in $\mathbb{C}^n$, *Acta Mathematica Scientia Ser. B,* **32**, 1675-1680 (2012)